Lecture 10 Feb 21

4.4.4 Local interpolation error

$(K, \mathcal{P}, \mathcal{N})$

K star-shaped, chunkiness γ

$\mathcal{P}_{m-1} \subset \mathcal{P} \subset W^m_\infty(K)$

$\mathcal{N} \subset (C^\ell(K))'$

order

$\begin{cases} m - \frac{n}{p} > \ell \\ p > 1 \end{cases}$ or $\begin{cases} m - \frac{n}{p} \geq \ell \\ p = 1 \end{cases}$ (so that $W^m_p(K) \subset C^\ell(K)$)

Then, for $i = 0, \dots, m$

$$|v - Iv|_{W^i_k(K)} \leq C_{m,n,\gamma,\sigma(\hat{K})} \operatorname{diam}(K)^{m-i} |v|_{W^m_p(K)}$$

shape only   size

$\hat{K} = \{ \hat{x} = \frac{x}{\operatorname{diam}(K)} : x \in K \}$, $\operatorname{diam}(\hat{K}) = 1$

$\sigma(K) = \| I \|_{C^\ell(K) \to W^m_p(K)}$

$Iv = \sum_{i=1}^{k} N_i(v) \phi_i$

reference element

$$K \xrightarrow[\text{affine eq.}]{\tilde{x} = F(x) = ax + b} \tilde{K}$$

$\mathcal{P}$, $\mathcal{N}$

$\tilde{\mathcal{P}} = (F^{-1})^*(\mathcal{P})$

$\tilde{\mathcal{N}} = F_*(\mathcal{N})$

$a \in GL(\mathbb{R}^n)$

4.4.11 Proposition

$$\sigma(\tilde{K}) \le C_{ref}\, \chi(a)$$

$$C_{ref} = C_{(K, \mathcal{P}, \mathcal{N})}$$

$$\chi(a) = \left(1 + \max|a_{ij}|\right)^{\ell} \left(1 + \max|(a^{-1})_{ij}|\right)^{m} |\det(a)|^{1/p}$$

Make assumptions on the mesh so that

$$\sigma(\hat{\tilde{K}}) \le C$$

i.e. $$\chi(\hat{\tilde{a}}) \le C$$

4.4.13 (with better notation)

$d = \mathrm{diam}(\Omega)$

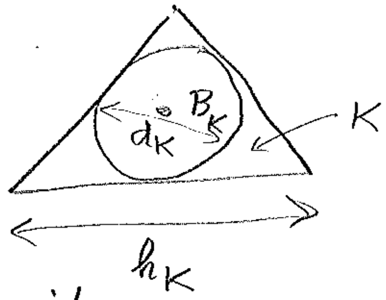
$\mathcal{F} = \{\mathcal{T}\}$ family of subdivisons of Ω
(not necessarily triangulations)

$h_K = \mathrm{diam}(K)$, $h_{\mathcal{T}} = \max\limits_{K \in \mathcal{T}} h_K / d \in (0,1]$

$d_K = \mathrm{diam}(B_K)$, $d_{\mathcal{T}} = \min\limits_{K \in \mathcal{T}} d_K$

B_K largest ball, star-shaped

$\gamma_K = 2 \dfrac{h_K}{d_K} \geq 2$ chunkiness



$\mathcal{F}$ is a non-degenerate family
(or regular or shape-regular)

if $\exists \rho > 0$:

$$0 < \rho \leq \frac{d_K}{h_K} \leq 1 \quad \forall K \in \mathcal{T} \ \forall \mathcal{T} \in \mathcal{F}$$

or $$\inf_{K \in \mathcal{T}} \frac{d_K}{h_K} \geq \rho > 0 \quad \forall \mathcal{T} \in \mathcal{F}$$

or $$\inf_{K \in \mathcal{T}} \frac{1}{\gamma_K} \geq \frac{\rho}{2} \quad \forall \mathcal{T} \in \mathcal{F}$$

or $$\sup_{\mathcal{T} \in \mathcal{F}} \sup_{K \in \mathcal{T}} \gamma_K \leq \frac{2}{\rho}$$

$\mathcal{F}$ is a <u>quasi-uniform</u> family

if $\exists\ \varrho > 0$:

$$\varrho \leq \frac{\min\limits_{K \in \mathcal{T}} d_K}{\max\limits_{K \in \mathcal{T}} h_K} \qquad \forall \mathcal{T} \in \mathcal{F}$$

or

$$(*)\ \ \varrho\, d \leq \frac{d_{\mathcal{T}}}{h_{\mathcal{T}}} \qquad \forall \mathcal{T} \in \mathcal{F}$$

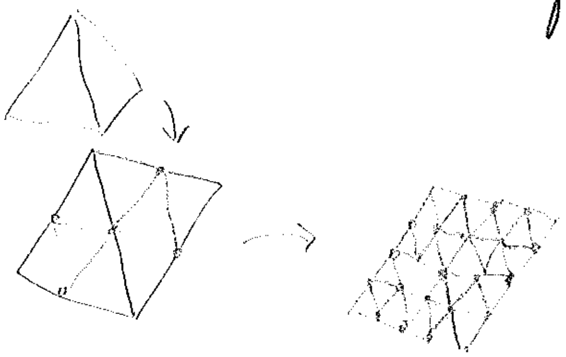
(all K in $\mathcal{T}$ are almost equal in shape and size and all K in all $\mathcal{T}$ are almost equal in shape)

Clearly $d_K \geq d_{\mathcal{T}}$, $h_K \leq \max h_K = d\, h_{\mathcal{T}}$

so $(*)$ implies

$$\frac{d_K}{h_K} \geq \frac{d_{\mathcal{T}}}{d\, h_{\mathcal{T}}} \geq \varrho$$

Therefore: quasi-uniform family $\Rightarrow$ regular family



quasi-uniform family of triangulations

Moreover, (*) implies

$$\frac{\min_{K\in T} h_K}{\max_{K\in T} h_K} \geq \frac{\min_T d_K}{\max_T h_K} = \frac{d_T}{d\, h_T} \geq \rho$$

i.e. same size in T

For a finite family $\mathcal{F}$ such a ρ always exists but may be very small if the $\mathcal{F}$ is "bad". Then ρ is a quantitative measure of how <u>regular</u> the family is.

Brenner Scott has $\mathcal{F} = \{T^h\}_{h\in(0,1]}$ where T^h is any subdivision such that

$$h_{T^h} \leq h$$

or

$$\max_{K\in T^h} h_K \leq h\, d.$$

4.4.20 Global interpolation error

Ω polyhedral

$\{\mathcal{T}^h\}$ non-deg. family of subdivisions

$(K, \mathcal{P}, \mathcal{N})$ reference element, ℓ, m, p

For all $T \in \mathcal{T}^h$ let

$(T, \mathcal{P}_T, \mathcal{N}_T)$ be affine eq to $(K, \mathcal{P}, \mathcal{N})$.

Then, for $s = 0, \dots, m$,

$$\left(\sum_{T \in \mathcal{T}^h} \| v - I^h v \|^p_{W^s_p(T)} \right)^{\frac{1}{p}} \leq C h^{m-s} |v|_{W^m_p(\Omega)}$$

and, for $s = 0, \dots, \ell$,

$$\max_{T \in \mathcal{T}_h} \| v - I^h v \|_{W^s_\infty(T)} \leq C h^{m-s-n/p} |v|_{W^m_p(\Omega)}.$$

$C = C(m, n, (K, \mathcal{P}, \mathcal{N}), p)$

The local interpol. error constant is

Proof $C_{m,n,\gamma_T,\sigma(\hat{T})}$. We have $\gamma_T \le \frac{2}{\rho}$.

Must show

$$\sup_{\mathcal{T}^h} \sup_{T \in \mathcal{T}^h} \sigma(\hat{T}) \le C(m, n, p, (K,P,N), \rho)$$

$$(\hat{T}, P_{\hat{T}}, N_{\hat{T}}) \cong (T, P_T, N_T) \cong (K, P, N)$$

$F(x) = ax + b$

Show $\chi(a) \le C$.

Show $a \in$ compact subset of $GL(\mathbb{R}^n)$

Non-degenerate family: $\exists B \subset \hat{T}$

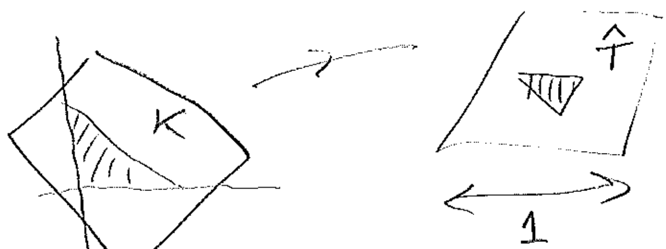
$$d_B = \frac{d_B}{h_{\hat{T}}} \ge \rho > 0$$

$$c\rho^m \le |B| \le |\hat{T}| = \int_{\hat{T}} d\hat{x} = |\det(a)| \int_T dx = c\,|\det(a)|$$

$|\det(a)| \ge c > 0$

upper bound for a?

May assume K contains a simplex



$S = \{x : \sum x_i \le t_0,\ x_i \ge 0\}$,

$F(te^i) = a(te^i) + b = t a e^i + b \in \hat{T}$, $0 \le t \le t_0$

$t_0 \| a e^i \| \le \operatorname{diam} \hat{T} = 1$

$\sum |a_{ij}| \le 1/t_0$

$\sum a_{kl} \underbrace{e^i_l}_{\delta_{il}} = a_{ki}$

Thus $a \in \{ \det(a) \geq c > 0,\ \max |a_{ij}| \leq C \}$

The proof is completed by using local interpol. error estimate.

For C^0 element we have in the global norm

$$|v - I^h v|_{W^s_p(\Omega)} = \left(\sum_T |v - I^h v|^p_{W^s_p(T)} \right)^{1/p} \leq$$

$$\leq C h^{m-s} |v|_{W^m_p(\Omega)}$$

for $s = 0, 1$.

(Note $I^h v$ is in $W^1_\infty(\Omega)$ but not more, (3.3.17).)