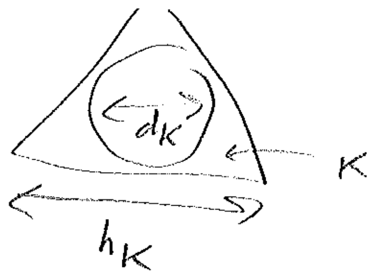
Lecture 11, Feb 24

4.5 Inverse estimates

$\mathcal{F} = \{\mathcal{T}^h\}$, $d = \mathrm{diam}(\Omega)$

h_K, $h_{\mathcal{T}^h} = \frac{1}{d} \max_{\mathcal{T}^h} h_K \le h$

d_K,



$\mathcal{F}$ is non-degenerate:

$$\frac{d_K}{h_K} \ge \rho \qquad \forall K \in \mathcal{T}^h, \ \forall \mathcal{T}^h \in \mathcal{F}.$$

$\mathcal{F}$ is quasi-uniform:

$$d_K \ge \rho \, d \, h$$

Scaling:

$$\hat{K} = \{ \hat{x} = h_K^{-1} x : x \in K \}, \quad h_{\hat{K}} = 1$$

$$\hat{v}(\hat{x}) = v(h_K \hat{x})$$

$$|\hat{v}|_{W^k_r(\hat{K})} = h_K^{k-n/r} |v|_{W^k_r(K)}$$

$$\hat{\mathcal{P}} = \{ \hat{v} : v \in \mathcal{P} \}$$

4.5.3 Local inverse inequality

Assume:

$\rho h \le h_K \le h, \quad h \in (0,1]$

$\mathcal{P} \subset W^{\ell}_{p}(K) \cap W^{m}_{q}(K), \quad 1 \le p,q \le \infty, \quad 0 \le m \le \ell$

$\dim(\mathcal{P}) < \infty$

Then

$$\|v\|_{W^{\ell}_{p}(K)} \le C\, h^{\overbrace{m-\ell}^{\le 0} + \frac{n}{p} - \frac{n}{q}} \|v\|_{W^{m}_{q}(K)}$$

$$\forall\, v \in \mathcal{P}$$

where $C = C(\hat{\mathcal{P}}, \hat{K}, \ell, p, q, \rho)$.

Note: "forward" inequality $\|v\|_{W^m_p(K)} \le \|v\|_{W^{\ell}_p(K)}$; $m \le \ell$.

Proof $C = C(\hat{\mathcal{P}}, \hat{K}, \ell, p, q, \rho)$.

$\underline{m=0}$

$\dim(\mathcal{P}) < \infty \Rightarrow \|\hat{v}\|_{W^j_p(\hat{K})} \le C \|\hat{v}\|_{L_q(\hat{K})} \qquad C = C(\hat{\mathcal{P}})$

$\|v\|_{W^{\ell}_p(K)} = \left(\sum_{j=1}^{\ell} |v|^p_{W^j_p(K)} \right)^{1/p}$, so $|\hat{v}|_{W^j_p(\hat{K})} \le C \|\hat{v}\|_{L_q(\hat{K})}$

$$|v|_{W^j_p(K)} = h_K^{-j+\frac{n}{p}} |\hat{v}|_{W^j_p(\hat{K})} \le C\, h_K^{-j+\frac{n}{p}} \|\hat{v}\|_{L_q(\hat{K})} =$$

$$= C\, h_K^{-j+\frac{n}{p}}\, h_K^{-\frac{n}{q}} \|v\|_{L_q(K)} = C \underbrace{h_K^{\overbrace{\ell-j}^{\ge 0}}}_{\le 1} \underbrace{h_K^{-\ell+\frac{n}{p}-\frac{n}{q}}}_{\le C h^{-\ell+\frac{n}{p}-\frac{n}{q}}} \|v\|_{L_p(K)}$$

Thus

$$\| v \|_{W^\ell(K)} \le C\, h^{-\ell + \frac{m}{p} - \frac{m}{q}} \| v \|_{L_q(K)}.$$

$\underline{0 < m \le \ell}$: Apply this to $D^\alpha v = D^\beta D^\gamma v$ in a clever way.

4.5.11 Global inverse inequality

Assume: $\{\mathcal{T}^h\}_{h \in (0,1]}$ quasi-uniform with parameter ρ.

$(K, \mathcal{P}, \mathcal{N})$ reference element

$\mathcal{P} \subset W^\ell_p(K) \cap W^m_q(K)$, $1 \le p, q \le \infty$, $0 \le m \le \ell$.

For $T \in \mathcal{T}^h$ let $(T, \mathcal{P}_T, \mathcal{N}_T) \cong (K, \mathcal{P}, \mathcal{N})$

$V^h = \{ v : v|_T \in \mathcal{P}_T \ \forall T \in \mathcal{T}^h \}$

Then, $C = C(\ell, p, q, \rho)$

$$\left(\sum_T \| v \|^p_{W^\ell_p(T)} \right)^{1/p} \le C h^{\overbrace{m - \ell}^{\le 0} + \overbrace{\min(0, \frac{m}{p} - \frac{m}{q})}^{\le 0}} \left(\sum_T \| v \|^q_{W^m_q(T)} \right)^{\frac{1}{q}}$$

$\forall v \in V^h$

<u>Proof</u> Quasi-uniformity $\Rightarrow$

$$hd \geq h_T \geq d_T \geq \rho d h \quad \forall T \in \mathcal{T}^h, \forall \mathcal{T}^h$$

So 4.5.3 applies:

$$\|v\|_{W^l_p(T)} \leq \underbrace{C(\hat{\mathcal{P}}_T, \hat{T}, l, p, q, m, \rho)}_{} h^{m-l+\frac{n}{p}-\frac{n}{q}} \|v\|_{W^m_q(T)}$$

$$\forall T \in \mathcal{T}^h, \forall v \in \mathcal{P}_T.$$

Non-degenerate $\Rightarrow$ $C(\hat{\mathcal{P}}_f, \hat{T}, l, p, m, \rho) \leq C(l, p, q, m, \rho)$

$p = \infty$ $\displaystyle \max_T \|v\|_{W^l_\infty(T)} \leq C h^{m-l+\frac{n}{p}-\frac{n}{q}} \max_T \|v\|_{W^m_q(T)} \leq C \left(\sum_T \|v\|^q_{W^m_q(T)}\right)^{\frac{1}{q}}$

$1 \leq p < \infty$:

$$\left(\sum_T \|v\|^p_{W^l_p(T)}\right)^{\frac{1}{p}} \leq C h^{m-l+\frac{n}{p}-\frac{n}{q}} \left(\sum_T \|v\|^p_{W^m_q(T)}\right)^{\frac{1}{p}}$$

$p \leq q$: $\displaystyle \left(\sum_T \|v\|^p_{W^m_q(T)}\right)^{\frac{1}{p}} \leq \left(\sum \|v\|^q_{W^m_q(T)}\right)^{\frac{1}{q}}$

$q \leq p$: $\displaystyle \left(\sum_T \|v\|^p_{W^m_q}\right)^{\frac{1}{p}} \leq \left(\sum 1^{r'}\right)^{\frac{1}{pr'}} \left(\sum \|v\|^{pr}_{W^m_q}\right)^{\frac{1}{pr}}$

$pr = q$

$\frac{1}{r'} = 1 - \frac{1}{r} = 1 - \frac{p}{q}$

$$= \underbrace{\left(\sum 1\right)}_{\leq Ch^{-n}}^{\overbrace{\frac{1}{p}-\frac{1}{q}}^{\geq 0}} \left(\sum \|v\|^q_{W^m_q}\right)^{\frac{1}{q}}$$

$$|\Omega| = \sum_T |T| \geq \sum_T |B_T| \geq C \sum d_T^n \geq C \sum (d\rho h)^n = C \rho^n h^n d^n \sum 1$$