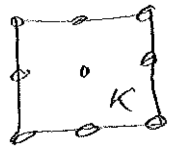
Lecture 12 Feb 27

4.6 Tensor product elements

$$P = Q_{m-1} = \underbrace{P^1_{m-1} \otimes \cdots \otimes P^1_{m-1}}_{n \text{ times}}$$



$$P_{m-1} \subset Q_{m-1} \subset P_{n(m-1)}$$

$$\mathcal{N} \subset (C^0(K))' \quad (l = 0)$$

Thm 4.4.4: $\left(m - \frac{n}{p} > 0, \ \geq 0, p = 1 \right)$

$$|v - Iv|_{W^k_p(K)} \leq C h_K^{m-k} |v|_{W^m_p(K)}$$

$k = 0, \dots, m$

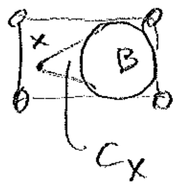
What is the effect of the extra terms in $P_{n(m-1)} \setminus P_{m-1}$?

This term is not optimal.

Note: $|v|_{W^m_p(K)} = 0$ for $v \in P_{m-1}$.

Special case

bilinear



$n=2, \quad m=2$

$Q_1 = \text{span}\{1, x_1, x_2, x_1x_2\}$

$\quad = \text{span}\{x^{(0,0)}, x^{(1,0)}, x^{(0,1)}, x^{(1,1)}\} = \text{span}_{\beta\in A^0}\{x^\beta\}$

$P_1 = \text{span}\{1, x_1, x_2\} = \text{span}_{A^1}\{x^\beta\}$

$P_2 = \text{span}\{1, x_1\, x_2, x_1x_2, x_1^2, x_2^2\} = \text{span}_{A^2}\{x^\beta\}$

$P_1 \subset Q_1 \subset P_2$

$A = \{(2,0), (0,2)\}, \qquad A^1 = \{(0,0), (1,0), (0,1)\}$

$A^0 = \{(0,0), (1,0), (0,1), (1,1)\}$

$A^2 = A^0 \cup A$

$$Q^3 u = \sum_{\alpha\in A^2} \int_B \frac{1}{\alpha!} D^\alpha u(y)(x-y)^\alpha \phi(y)\,dy$$

$$Q^A u = \sum_{\alpha\in A^0} \text{—— " ——}$$

$$u(x) = (Q^3u)(x) + (R^3u)(x)$$
$$= (Q^A u)(x) + \sum_{\alpha\in A^2\setminus A^0} \int_B \frac{1}{\alpha!} D^\alpha u(y)(x-y)^\alpha \phi(y)\,dy + R^3 u(x)$$
$$= Q^A u(x) + \sum_{\alpha\in A} \text{— " ——}$$
$$+ \sum_{|\alpha|=3} \underbrace{\int_0^1 \int_B \frac{3s^2}{\alpha!} D^\alpha u(x+s(y-x))(x-y)^\alpha \phi(y)\,dy\,ds}_{= \int_{C_x} k_\alpha(x,z) D^\alpha u(z)\,dz}$$

see (4.2.8)

$|\alpha|=3: A^3\setminus A^2 = \{(2,1), (1,2), (3,0), (0,3)\}$

We want to use only $\alpha\in A$.

Integrate by parts:

e.g. $\alpha = (3,0)$

$$\int_{C_x} k_{(3,0)}(x,z)\, D^{(3,0)} u(z)\, dz =$$

$$= \int_{C_x} \underbrace{-D_z^{(1,0)} k_{(3,0)}(x,z)}_{\tilde{k}_{(2,0)}(x,z)}\, D^{(2,0)} u(z)\, dz$$

$$|\tilde{k}_{(2,0)}(x,z)| = |D_z^{(1,0)} k_{(3,0)}(x,z)| \le$$

$$\le C\, |x-z|^{\underbrace{|(3,0)| - m}_{=2} - |(1,0)|} = C\,|x-z|^{2-m}$$

$$= C\,|x-z|^{|(2,0)| - m}$$

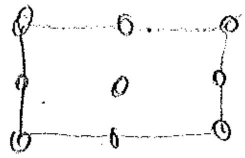
Thus:

$$u(x) - (Q^A u)(x) = \sum_{\alpha \in A} \int_{C_x} \tilde{k}_\alpha(x,z)\, D^\alpha u(z)\, dz$$

where
$$|\tilde{k}_\alpha(x,z)| \le C\, |x-z|^{|\alpha| - m}$$

This holds for Q_{m-1} in general.

<u>Special case</u> biquadratic



$n = 2, \; m = 3$

$P_2 \subset Q_2 \subset P_4$

$P_4 = \operatorname{span}_{A^4} \{ x^\beta \}$

$A^4 = \{ (0,0), (1,0), (0,1), (1,1), (2,0), (0,2), (2,1), (1,2), (3,0), (0,3)$

A^1

A^2

A^3

$(2,2), (3,1), (1,3), (4,0), (0,4) \}$

$A^0 = A^2 \cup \{ (2,1), (1,2), (2,2) \}$

$A = \{ (3,0), (0,3) \}$

$A^4 \setminus A^0 = \{ \underbrace{(3,0), (0,3)}_{\text{O.K.}}, \underbrace{(3,1), (1,3), (4,0), (0,4)}_{\text{integrate by parts}} \}$

$$u = Q^5 u + R^5 u$$

$$= Q^A u + \sum_{\alpha \in A^4 \setminus A^0} \int_B \frac{1}{\alpha!} D^\alpha u(y) (x-y)^\alpha \phi(y)\, dy$$

$$+ \sum_{|\alpha| = 5} \int_{C_x} k_\alpha(x, z) D^\alpha u(z)\, dz$$

From Lemma (4.3.6):

$$| u - Q^A u |_{W_p^k(K)} \le C_{m,n,\gamma}\, d^{m-k} \Big(\sum_{i=1}^{n} \Big\| \frac{\partial^m u}{\partial x_i^m} \Big\|_{L_p(K)}^p \Big)^{\frac{1}{p}}$$

$\alpha \in A$

Let $I : C^0(K) \to Q_{m-1}$ be the interpolator.

Then using ($m - \frac{n}{p} > 0$, ≥ 0 $p=1$)

$$\begin{cases} I Q^A v = Q^A v & (\text{because } Q^A u \in Q_{m-1}) \\ | I v |_{W_p^k(\hat{K})} \le \sigma(\hat{K}) \| v \|_{\substack{C^0(\hat{K}) \\ L_\infty(\hat{K})}} \le C\sigma(\hat{K}) \| v \|_{W_p^m(\hat{K})} \end{cases}$$

as in (4.4.4) we prove

$$| u - I u |_{W_p^k(\hat{K})} \le | u - Q^A u |_{W_p^k(\hat{K})} + | \underbrace{Q^A u - I u}_{= I(Q^A - u)} |_{W_p^k(\hat{K})}$$

$$\le \dots \le C \, \| u - Q^A u \|_{W_p^m(\hat{K})}$$

$$\le C \Big(\sum_{\alpha \in A} \| D^\alpha u \|_{L_p(\hat{K})}^p \Big)^{1/p}$$

This leads to

(4.6.11) Theorem $m - \frac{n}{p} \geq 0$

$$| u - I u |_{W_p^k(K)} \le C h_K^{m-k} \Big(\sum_{i=1}^{n} \Big\| \frac{\partial^m u}{\partial x_i^m} \Big\|_{L_p(K)}^p \Big)^{1/p}$$

$k = 0, \dots, m.$