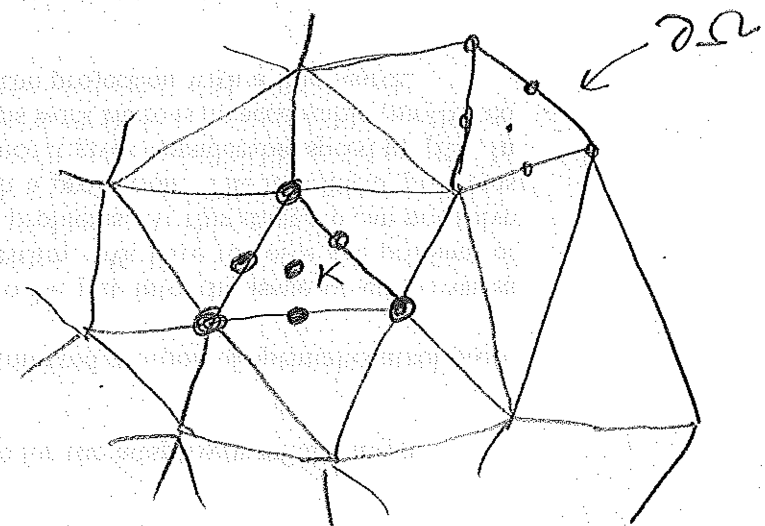
Lecture 14 March 2

4.8 The Scott-Zhang interpolator

For simplicity: consider a C^0 Lagrange finite element of order m (degree $m-1$) on a non-degenerate family of triangulations $\{T^h\}$ in $\Omega \subset \mathbb{R}^n$, $n = 2, 3$.

(K, P_K, N_K)



$P_K = P_{m-1}$

$N_K \subset (C^0(K))'$

$$\begin{cases} m - \frac{n}{p} > 0 & p > 1 \\ m - \frac{n}{p} \geq 0 & p = 1 \end{cases}$$

so that $W_p^m(K) \subset C^0(K)$.

and $N_i^K(u) = u(P_i)$ is defined for $u \in$ ~~$C^0(K)$~~ $W_p^m(K)$

Local interpolator:

$$(I^K u)(x) = \sum_{i=1}^{k} N_i^K \varphi_i^K(x) = \sum_{N \in N_K} N^K(u) \varphi_N^K(x)$$

Global interpolator:

$$(I^h u)(x) = \sum_z N_z(u) \phi_z(x), \quad u \in C(\bar{\Omega})$$

sum over all nodes of T^h

$N_z(u) = u(z)$

$$\phi_z(z') = \begin{cases} 1 & z' = z \\ 0 & z' \neq z \end{cases}$$

$V^h = \text{span}\{\phi_z\}$

What if $u \notin C(\bar{\Omega})$?

For example: $u \in H^1(\Omega)$,

We have
$\|I^h u\|_{W^1_p(\Omega)} \leq C \|u\|_{W^m_p(\Omega)}$
but sometimes we do not want to use $\|u\|_{W^m_p}$ ($m \geq 2$!) even if $u \in C^\infty(\Omega)$
$m \geq 2$.

We will define

$$(\tilde{I}^h u)(x) = \sum_z \tilde{N}_z(u)\, \phi_z(x)$$

where $\tilde{N}_z(u)$ is based on a local average of u..

Desirable properties:

* $\tilde{I}^h v = v \quad \forall v \in V^h$
* $\tilde{I}^h v|_{\partial\Omega} = 0$ if $v|_{\partial\Omega} = 0$
* $\|I^h v\|_{W^1_p(\Omega)} \leq C \|v\|_{W^1_p(\Omega)}$
* error estimate

For each node z:

choose K_z so that $z \in K_z$

and then a sub-simplex $\tilde{K}_z \subset K$.

i) $z \in \mathrm{int}(K_z)$ choose $\tilde{K} = K_z$

ii) $z \in \mathrm{int}(\partial K_z)$ ~~choose $\tilde{K} = \partial K$~~

that is z edge/face E of K_z,

choose $\tilde{K}_z = E$

iii) $z \in$ ~~corner~~ vertex of K_z

choose $\tilde{K}_z =$ one edge/face of K_z

emanating from z

(or $\tilde{K}_z = K_z$.)

iv) if $z \in \partial\Omega$ choose $\tilde{K}_z \subset \partial\Omega$.

Given $(K_z, P_z, \mathcal{N}_z)$ define $(\tilde{K}_z, \tilde{P}_z, \tilde{\mathcal{N}}_z)$

by $\tilde{P}_z = \{ f|_{\tilde{K}_z} : f \in P_z \}$

$$N_z^{\tilde{K}_z}(u) = u(z)$$

local nodal basis: $\{ \varphi_N^{\tilde{K}_z} \}_{N \in \tilde{\mathcal{N}}_z}$

$L_2(\tilde{K}_z)$ - dual basis $\{ \psi_N^{\tilde{K}_z} \}_{N \in \tilde{\mathcal{N}}_z}$:

$$(*) \qquad \int_{\tilde{K}_z} \phi_M \psi_N \, dx = \delta_{MN}$$

that is: by Riesz repr. thm.

if $N(u) = u(p)$, p node in $\tilde{K}_z$

$$(**) \qquad \forall N \in \tilde{\mathcal{N}}_z \in \tilde{P}_z' \ \exists! \ \psi_N \in \tilde{P}_z : \int_{\tilde{K}_z} \psi_N u \, dx = N(u) = u(p) \quad \forall u \in \tilde{P}_z$$

$\forall p$ node $\exists! \psi_{N_p}^{\tilde{K}_z}$: $u(p) = N_p(u) = (u, \psi_{N_p}^{\tilde{K}_z})$

Extend to $\{ \psi_N^{\tilde{K}_z} \}_{N \in \mathcal{N}_z}$ by $\psi_N^{\tilde{K}_z} = 0$ if $N \in \mathcal{N}_z \setminus \tilde{\mathcal{N}}_z$

Then $(*)$ holds for $N, M \in \mathcal{N}_z$.

Define global interpolator

$$(\tilde{I}^h u)(x) = \sum_z \left(\int_{\tilde{K}_z} \tilde{\psi}_{N_z}^{\tilde{K}_z}(x) \, u(x) \, dx \right) \phi_z(x)$$

$u \in L^1(\tilde{K}_z)$

where ψ_{N_z} is obtained by taking $p = z$ in $(**)$. $\forall z$

4.8.7 Thm

$$\tilde{I}_h v = v \quad \forall v \in V^h$$

Proof: $\tilde{I}^h_h v = \sum \underbrace{\int_{\tilde{K}_z} \psi_{N_z} v \, dx}_{(**) \; = N_z(v) = v(z)} \phi_z$

(4.8.8) $\quad v|_{\partial\Omega} = 0 \Rightarrow \tilde{I}^h v|_{\partial\Omega} = 0$

Error estimate.

Trace thm: (1.6.6)

$$\|v\|_{L_1(\tilde{K}_z)} \le C \|v\|_{W^1_p(K_z)}$$

$$\| \tilde{I}^h v \|_{W^m_p(K)} \le C \| v \|_{W^1_p(S_K)}$$

$0 \le s \le k \le m, \quad k \ge 1$

$$\| v - \tilde{I}^h v \|_{W^s_p(K)} \le \| v - Q^k v \|_{W^s_p} + \underbrace{\| \tilde{I}^h (v - Q^k v) \|_{W^s_p}}_{\le C \| v - Q^k v \|_{W^1_p(S_K)} \le C \| v - Q^k v \|_{W^s_p(S_K)}}$$

$$\le C \| v - Q^k v \|_{W^s_p(S_K)}$$

Bramble Hilbert

$$\le C h_K^{k-s} | v |_{W^k_p(S_K)}$$

$\# \{K\}_{K \in S_K} \le C$ by non-degeneracy