Lecture 4 Jan 23, 2012 3.1 – 3.2

Recall

$$u_h(x) = \sum_{i=1}^{N_h} \underbrace{u_h(x_i)}_{\substack{\text{node values}\\ \text{degrees of freedom}}} \underbrace{\phi_i(x)}_{\text{basis fcns}}$$

Recall Finite dim vector space V

Basis $\{e_k\}_{k=1}^{d}$. Every $v \in V$ uniquely written

$$v = \sum_{k=1}^{d} \alpha_k e_k$$

How to determine the coeff α_k?

~~There is a~~

Dual space: $V' = \{ L : V \to \mathbb{R} \text{ linear} \}$

Unique dual basis to $\{e_k\}$: $\{l_k\}_1^d \subset V'$

$$l_k(e_j) = \delta_{ij}$$

Then $\quad l_j(v) = \sum \alpha_k l_j(e_k) = \alpha_j$

Thus: $\quad v = \sum_{k=1}^{d} l_k(v) e_k$

3.1 The finite element

(3.1.1) Definition $(K, P, \mathcal{N})$ is a finite element if

(i) $K \subset \mathbb{R}^n$, bdd, closed, $\operatorname{int}(K) \neq \emptyset$, ∂K p.w. smooth (element domain)

(ii) P finite dim space of fcns on K (shape functions)

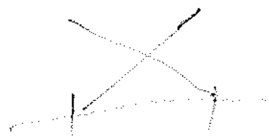
(iii) $\mathcal{N} = \{N_1, \dots, N_d\}$ a basis for P' (nodal variables or degrees of freedom)

(3.1.2) Def. The basis $\{\phi_i\}_1^d \subset P$ dual to $\mathcal{N}$ is the nodal basis of P.

Ex 1-D Lagrange element

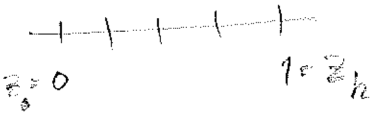
$K = [0,1]$, $P = \mathcal{P}_1 = \{\text{poly degree} \le 1\}$

$\mathcal{N} = \{N_1, N_2\}$, $N_1(v) = v(0)$, $N_2(v) = v(1)$.



$\phi_1(x) = 1 - x$, $\phi_2(x) = x$

$P = \mathcal{P}_k$, $\mathcal{N} = \{N_0, \dots, N_k\}$



3.1.4 Lemma Let $\dim(P) = d$, $(a) \Leftrightarrow (b)$

(a) $\mathcal{N} = \{N_1, \dots, N_d\}$ ~~dual~~ basis for P'

(b) $\begin{cases} v \in P \\ N_i(v) = 0 \end{cases} \Rightarrow v = 0$

Proof Let $\{\phi_i\}_1^d$ basis for P.

(a) $\Leftrightarrow \mathcal{N}$ is basis for $P' \Leftrightarrow \forall L \in P', \exists! \{\alpha_j\}\ L = \sum_1^d \alpha_j N_j$

$\Leftrightarrow \forall y_i\ \exists!\ \alpha_j : y_i := L(\phi_i) = \sum \alpha_j N_j(\phi_i)$

$\Leftrightarrow B\alpha = y$ solvable

$\Leftrightarrow B$ invertible

$v \in P,\ v = \sum \beta_j \phi_j,\ N_i(v) = \sum \beta_j N_i(\phi_j) = 0$

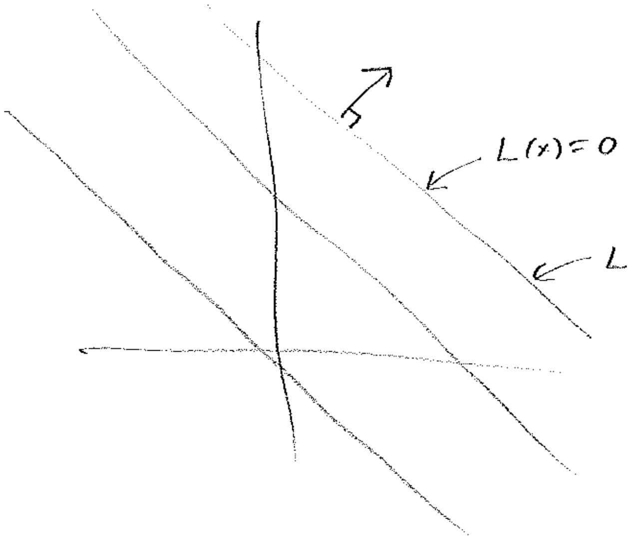
$C\beta = 0$

(b) $\Leftrightarrow$ "$C\beta = 0 \Rightarrow \beta = 0$" $\Leftrightarrow C$ inv.

But $C = B^T$ so $a \Leftrightarrow b$

3.1.8 Def $\mathcal{N}$ determines $\mathcal{P}$ if $\begin{cases} \psi \in \mathcal{P} \\ N(\psi) = 0 \ \forall N \in \mathcal{N} \end{cases} \Rightarrow \psi = 0.$

Hyperplane in $\mathbb{R}^n$:



$L(x) = n \cdot x + n_0$

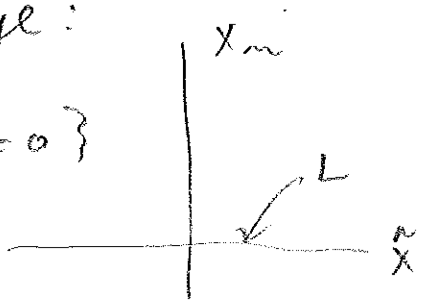
$L = \{ x \in \mathbb{R}^n : L(x) = 0 \}$

3.1.10 Lemma

Let $P \in \mathcal{P}_d$ $(d \geq 1)$ with $P = 0$ on a hyperplane L.

Then $P = LQ$ with $Q \in \mathcal{P}_{d-1}$.

Proof Affine coord change:



$L(x) = L(\hat{x}, x_n) = x_n$ so $L = \{ x : x_n = 0 \}$

$$P(x) = P(\hat{x}, x_n) = \sum_{j=0}^{d} \sum_{|i| \leq d-j} c_{ij} \hat{x}^i x_n^j$$

$$0 = P(\hat{x}, 0) = \sum_{|i| \leq d} c_{i0} \hat{x}^i \Rightarrow c_{i0} = 0$$

i.e. no x_n^0 terms in P.

Then $P = x_n \sum \sum c_{ij} \hat{x}^i x_n^{j-1}$

3.2 Triangular finite elements

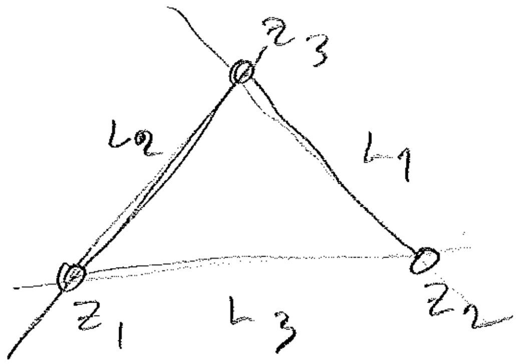
Now $n = 2$.

$P_1 = \{ a_0 + a_1 x_1 + a_2 x_2 \}$

$P_2 = \{ \quad " \quad - \quad + a_{11} x_1^2 + a_{12} x_1 x_2 + a_{22} x_2^2 \}$

k	$\dim(P_k)$
1	3
2	6
3	10
k	$\frac{1}{2}(k+1)(k+2)$

Lagrange triangle



Ex $\quad k = 1 \qquad P = P_1$

$\mathcal{N} = \{N_1\ N_2 N_3\}$

$N_i(v) = v(z_i)$

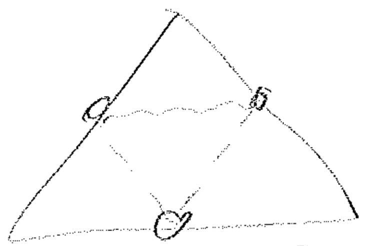
Show $\mathcal{N}$ det P.

$\begin{cases} p \in P \\ N_i(v) = 0 \end{cases}$ Show $p = 0$.

$\begin{cases} P(z_2) = P(z_3) = 0 \Rightarrow P|_{L_1} = 0 \\ P|_{L_1} \in P_1 \end{cases}$

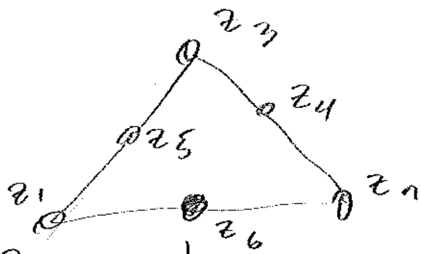
So $P = L_1 Q = c L_1$

$0 = P(z_1) = c L_1(z_1) \Rightarrow c = 0$.



Crouzeix-Raviart

$k = 2 \quad P = P_2$



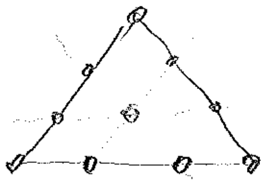
$P = 0$ on L_1

$P = L_1 Q_1$

$Q_1 = 0$ on $L_2 \qquad Q_1 = L_2 Q_2 = c L_2$

$P = c L_1 L_2$

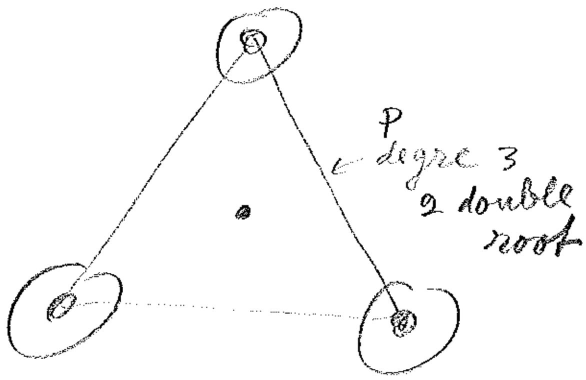
$k = 3$



Hermite

$k = 3$

$d = 10$



$P = c L_1 L_2 L_3$