Lecture 4 Jan 24, 2012

3.3 The interpolant

(3.3.1) Def. Let $(K, P, \mathcal{N})$ be a finite element and $\{\phi_i\}_1^k$ the nodal basis. The local interpolator I_K is defined by

$$I_K v = \sum_{i=1}^{k} N_i(v)\,\phi_i$$

for fcns v for which $N_i(v)$ are defined.

(3.3.4) – (3.3.7) Prop.

- I_K is a linear operator
- $N_i(I_K v) = N_i(v)$
- $I_K(v) = v$ if $v \in P$
- $I_K^2 = I_K$

Straightforward proofs.

(3.3.8) $\{K_i\}_{i=1}^{N}$ is a <u>subdivision</u> of Ω if
K_i are element domains with
(1) $\mathrm{int}(K_i) \cap \mathrm{int}(K_j) = \phi$ for $i \neq j$
(2) $\bigcup_{i=1}^{N} K_i = \overline{\Omega}$

(3.3.9) Let $\mathcal{T} = \{K\}$ be a subdivision of Ω such that there is a finite element (K, P, N) for each $K \in \mathcal{T}$. Let m be the highest order of derivative in the nodal variables. Then for $f \in C^m(\overline{\Omega})$ we define the global interpolant $I_{\mathcal{T}} f$ by

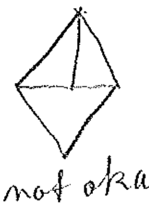
$$I_{\mathcal{T}} f|_K = I_K f \quad \forall K \in \mathcal{T}.$$

The interpolant is not cont^s unless we assume

(3.3.11) A <u>triangulation</u> of a polygonal domain $\Omega \subset \mathbb{R}^2$ is a subdivision $\mathcal{T} = \{K\}$ where K are triangles such that

(3) no vertex of any triangle is an interior point of any edge of another triangle


okay


not okay

(3.3.15) If $I_T f \in C^r$ $\forall f \in C^m(\bar{\Omega})$ then I_T has cont. order r. The $V_T = \{ I_T f : f \in C^m(\bar{\Omega}) \}$ is a C^r finite element space.

An element $(K, P, \mathcal{N})$ is called a C^r element if it can be used to construct a C^r finite elent space.

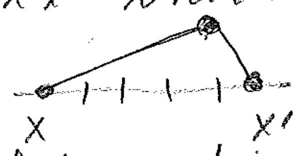
To do so we must place the nodes carefully.

(3.3.17) The Lagrange and Hermite elements are C^0 elements and Argyris is C^1. This means that for any triangulation T of Ω we can place nodes in the elements $(K, P_k, \mathcal{N})$, $K \in T$, so that $I_T f \in C^r$ for $f \in C^m(\bar{\Omega})$.

$r = \begin{cases} 0 & \text{lagrange \& hermite} \\ 1 & \text{argyris} \end{cases}$

$m = \begin{cases} 0 & \text{lag.} \\ 1 & \text{herm} \\ 2 & \text{arg.} \end{cases}$

For example, for an edge $\overline{XX'}$ choose edge nodes $\xi_i (X' - X) + X$ where



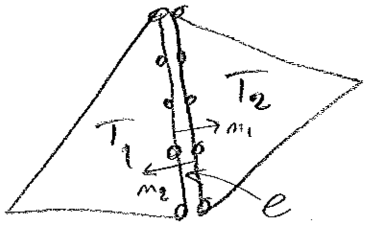
$\{\xi_i : 1, \dots, k-1-2m\}$ are fixed and symmetric around $\xi = \frac{1}{2}$

Moreover, $I_T f \in W^{r+1}_{\infty}(\Omega)$.

[This proof is not complete in the book.]

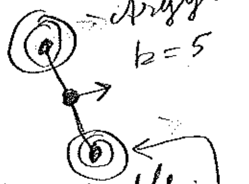
Proof Let T_1, T_2 have a common edge e.

We have chosen edge nodes so that they coincide.



Let $w = I_{T_1} f - I_{T_2} f$ (poly degree k)

with $w|_e$ nodal values $= 0$.



Hence $w|_e = 0$.

Thus I_T is cont^s. For Argyris: see next page $\left\{ \begin{array}{l} \frac{\partial w}{\partial e}\Big|_e = 0 \\ \frac{\partial w}{\partial n}\Big|_e \in P_4 \text{ with 2 double roots, 1 simple} \end{array} \right.$

Compute weak derivative $D^\alpha I_T f$, $|\alpha| = r+1$.

$\frac{\partial}{\partial e}\frac{\partial w}{\partial n}\Big|_e = 0$ because $D^2 w = 0$

Take $\phi \in C_0^\infty(\Omega)$. $|\alpha| = 1$ so that $D^\alpha = \frac{\partial}{\partial x_i}$

$$D^\alpha I_T f\,(\phi) = (-1)^{|\alpha|} \int_\Omega I_T f \, D^\alpha \phi \, dx = (-1)^{|\alpha|} \sum \int I_T f \underbrace{D^\alpha \phi}_{=\frac{\partial \phi}{\partial x_i}} dx$$

$$= - \sum_T \int_{\partial T} n_i I_T f \, \phi \, dS + \sum_T \int_T \frac{\partial}{\partial x_i} I_T f \, \phi \, dx$$

$\{ n_2 = -n_1 \}$

$$= - \sum_e \int_e n_{1,i} \underbrace{\left(I_{T_1} f - I_{T_2} f\right)}_{=0} \phi \, dS + \sum_T \int_T D^\alpha I_T f \, \phi \, dx$$

Note: boundary edges do not appear, because $\phi = 0$ near $\partial\Omega$.

Thus $D^\alpha I_T f|_T = D^\alpha I_T f$, piecewise derivative.

Clearly $D^\alpha I_T f \in L_\infty$, so that $I_T f \in W^1_\infty(\Omega)$.

Similarly for Argyris $|\alpha| = 2$:

use continuity of $\frac{\partial I_T f}{\partial x_i}$ on e.

For Argyris we also must show

$Dw|_e = 0$, that is

$\frac{\partial w}{\partial e}\Big|_e = 0$ (this clear because $w|_e = 0$)

and $\frac{\partial w}{\partial n}\Big|_e = 0$. Let $k = 5$.

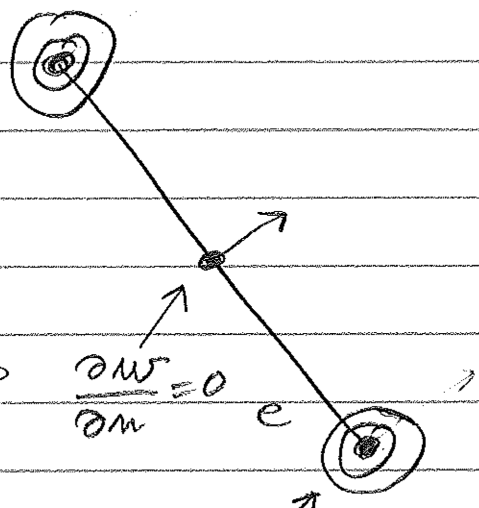
But $\frac{\partial w}{\partial n}\Big|_e$ is poly of

degree 4 with

1 simple root

and 2 double roots.

Hence $\frac{\partial w}{\partial n}\Big|_e = 0$.

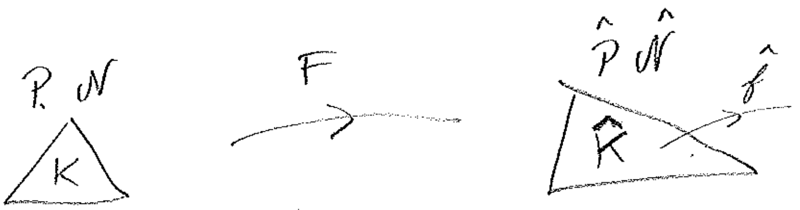


$\frac{\partial w}{\partial n} = 0$

$\frac{\partial}{\partial e}\frac{\partial w}{\partial n} = 0$

because $Dw = 0$ and $D^2 w = 0$ here.

3.4 equivalence of elements



$F(x) = Ax + b$
A non-singular
$\hat{x} = F(x)$

push-forward $F_*: P' \to \hat{P}'$, $N \mapsto F_*(N) = N(F^*(\cdot))$ $\qquad F_*(N)(\hat{f}) = N(F^*(\hat{f})) = N(\hat{f} \circ F)$

pull-back $F^*: \hat{P} \to P$, $F^*(\hat{f}) = \hat{f} \circ F$
$\hat{f} \mapsto \hat{f} \circ F$

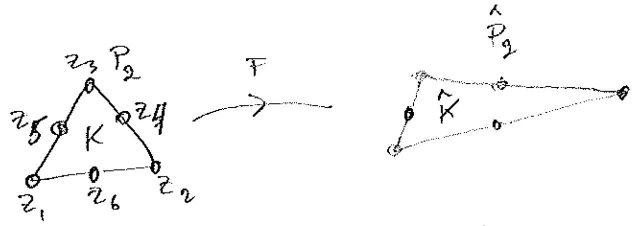
(3.4.1) (K, P, N) is affine eq to $(\hat{K}, \hat{P}, \hat{N})$ if

(i) $F(K) = \hat{K}$

(ii) $F^*(\hat{P}) = P$

(iii) $F_*(N) = \hat{N}$

Lagrange



$\hat{x} = Ax + b$, $\quad x = A^{-1}\hat{x} - A^{-1}b$

$f(x) = F^*(\hat{f})(x) = \hat{f}(F(x)) = \hat{f}(Ax + b)$ $\qquad z_5 = \frac{1}{2}(z_1 + z_3)$

$\hat{f}(\hat{x}) = f(A^{-1}\hat{x} - A^{-1}b)$ $\qquad \hat{z}_5 = F(z_5) = \frac{1}{2}F(z_1) + \frac{1}{2}F(z_3)$

$\qquad\qquad = \frac{1}{2}(\hat{z}_1 + \hat{z}_3)$

$\hat{f} \in \hat{P}_2 \iff f \in P_2$

$\hat{N}_i = F_*(N_i) = N_i(F^*(\cdot))$

$F_*(N)(\hat{f}) = N_i(\hat{f} \circ F) = (\hat{f} \circ F)(z_i) = \hat{f}(F(z_i)) = \hat{f}(\hat{z}_i) = \hat{N}_i(\hat{f})$