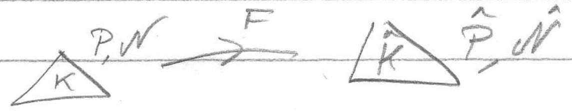
Lecture 5 Jan 27, 2012

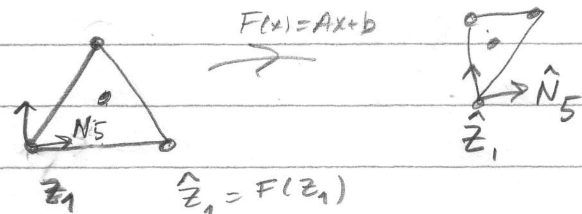


3.4 continued

$(K, P, \mathcal{N})$ and $(\hat{K}, \hat{P}, \hat{\mathcal{N}})$ are affine equivalent if $\exists$ affine mapping F so that

$$F(K) = \hat{K} \qquad \hat{z} = F(z) \in \hat{K} \Leftrightarrow z \in K$$
$$F^*(\hat{P}) = P \qquad f = F^*(\hat{f}) = \hat{f}\circ F \in P \Leftrightarrow \hat{f} \in \hat{P}$$
$$F_*(\mathcal{N}) = \hat{\mathcal{N}} \qquad F_*(N_i) = N_i(F^*\hat{f}) = \hat{N}_i(\hat{f})$$

<u>Example</u> Hermite $k=3$



Let $N_5(f) = \frac{\partial f}{\partial x_1}(z_1)$

$\hat{z}_1 = F(z_1)$

$$F_*(N_5)(\hat{f}) = N_5(\overbrace{\hat{f}\circ F}^{=F^*(\hat{f})}) = \frac{\partial}{\partial x_1}(\hat{f}\circ F)(z_1) = \sum_i \frac{\partial \hat{f}}{\partial \hat{x}_i}(\hat{z}_1)\underbrace{\frac{\partial \hat{x}_i}{\partial x_1}}_{=\frac{\partial F_i}{\partial x_1}(z_1) = A_{i1}} =$$

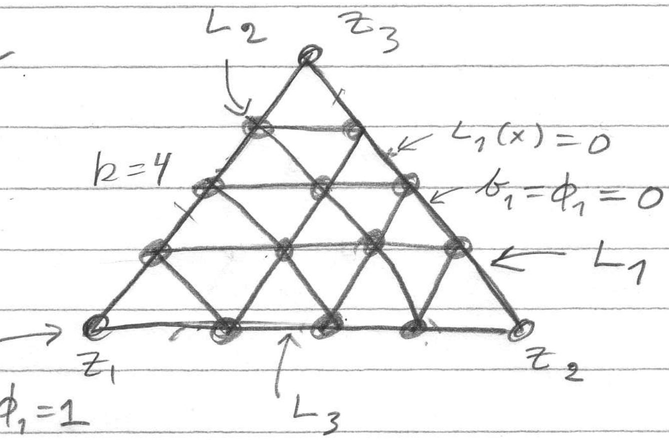
$$= \sum_i \frac{\partial \hat{f}}{\partial \hat{x}_i}(\hat{z}_1) A_{i1} \neq \hat{N}_5(\hat{f}) = \frac{\partial \hat{f}}{\partial \hat{x}_1}(\hat{z}_1)$$

Not affine equivalent.

(3.4.5) <u>Theorem</u> There exist nodal placements so that all Lagrange elements of degree k are affine eq.

<u>Proof</u> Use barycentric coordinates to place nodes in a coordinate-free way.

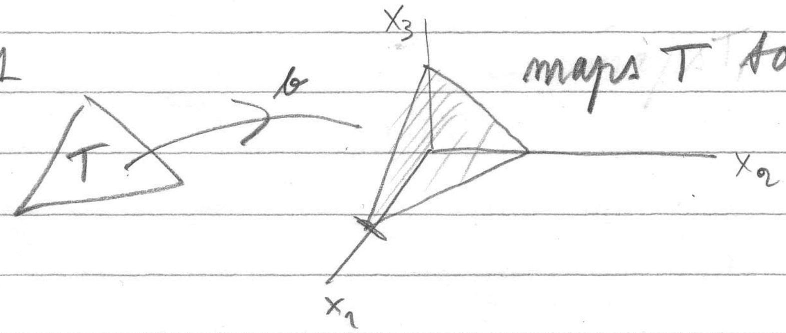
Barycentric coordinates $\{b_i\}_{i=1}^3$ are equal to $\{\phi_i\}_{i=1}^3$, the nodal basis for the P_1 element, or the L_i normalized so that $L_i = 1$ in the opposite vertex. $b_1 = \phi_1 = 1$



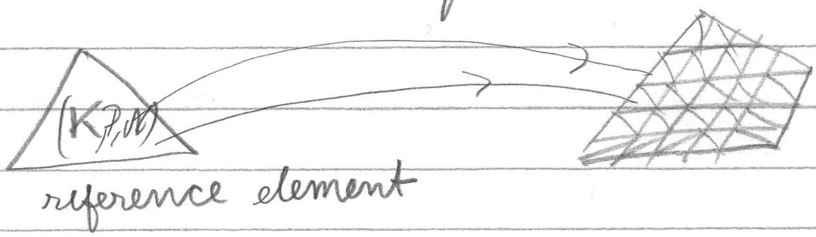
$$\mathbb{R}^2 \to \mathbb{R}^3$$
$$(x,y) \longmapsto b(x,y) = \big(b_1(x,y),\, b_2(x,y),\, b_3(x,y)\big)$$

$$\sum_{i=1}^{3} b_i = 1$$

maps T to triangle in $\mathbb{R}^3$



One element maps to every element in the triangulation.



(the same K, P)

(3.4.6)-(3.4.7) $(K, P, \mathcal{N})$ and $(K, P, \tilde{\mathcal{N}})$ are <u>interpolation</u> equivalent if

$$I_{\mathcal{N}} f = I_{\tilde{\mathcal{N}}} f \quad \forall f$$

This holds iff $\mathcal{N}$ is a linear combination of $\tilde{\mathcal{N}}$.

This applies to Hermite

(3.4.9) If $(K, P, \mathcal{N}) \overset{\text{affine}}{\cong} (\hat{K}, \hat{P}, \hat{\mathcal{N}}) \overset{\text{interpol}}{\cong} (\hat{K}, \hat{P}, \tilde{\mathcal{N}})$

then $(K, P, \mathcal{N})$ and $(\hat{K}, \hat{P}, \tilde{\mathcal{N}})$ are

<u>affine-interpolation</u> eq

3.5 Rectangular elements

Tensor product of 2-variables

$$Q_k = P_k^1 \otimes P_k^1 = \left\{ \sum_j p_j(x) q_j(y) : p_j, q_j \text{ poly degree} \le k \right.$$

in 1 variable $\}$

$\cancel{Q_1 = \{ a_0 + a_{10}x + a_{01}y + a_{11}}$

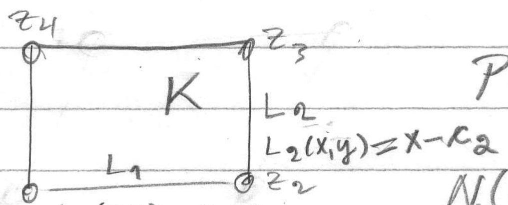
$\cancel{Q_2 = \{ a_0 + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2}$

$$Q_1 = \{ \underbrace{a_{00} + a_{10}x + a_{01}y}_{= P_1} + a_{11}xy \} \subset P_2$$

$$Q_2 = \{ \underbrace{\text{———} \quad '' \quad \text{———} + a_{20}x^2 + a_{02}y^2}_{= P_2} + a_{21}x^2y + a_{12}xy^2 + a_{22}x^2y^2 \} \subset P_4$$

$$P_k \subset Q_k \subset P_{2k}, \quad \dim(Q_k) = (P_k^1)^2$$

$k = 1$



$P = Q_1$, N as in the fig.

$N_i(\phi) = 0 \qquad \phi(x,y) = \sum p_i(x) q_i(y)$ on L_1 ($q_i(y) = c_1$)

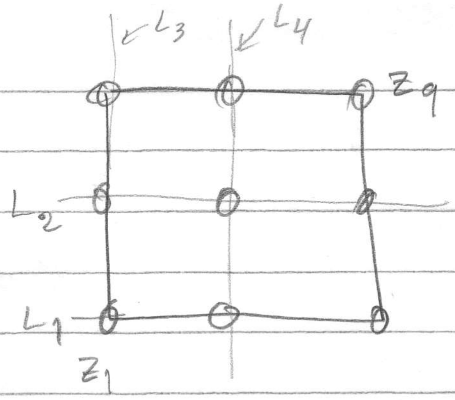
$\phi|_{L_1}$ = poly degree 1, two roots $\Rightarrow \phi|_{L_1} = 0 \Rightarrow \phi = L_1 Q_1$

$Q_1|_{L_2}$ = poly degree, 2 roots $\Rightarrow Q_1|_{L_2} = 0 \Rightarrow Q_1 = c L_2$

So $\phi(x,y) = c\, L_1(y) L_2(x)$

$$0 = \phi(z_4) = c\, \underbrace{L_1(z_4)}_{\neq 0}\, \underbrace{L_2(z_4)}_{\neq 0} \Rightarrow c = 0 \Rightarrow \phi = 0$$

$k=2$



$d=9$

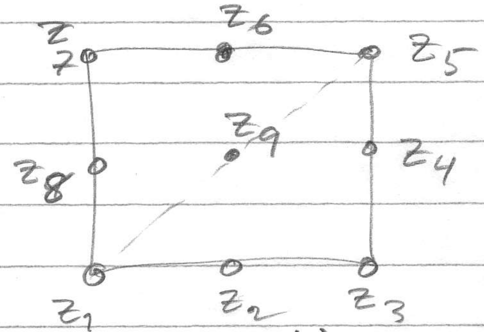
$N_i(\phi)=0 \Rightarrow \phi = c L_1 L_2 L_3 L_4$

$0 = \phi(z_9) = c L \dots L \Rightarrow c = 0$

The serendipity element $(k=2)$

Lemma $\exists c_i : \sum_{i=1}^{8} c_i \phi(z_i) = \phi(z_9) \ \forall \phi \in P_2$

(same c_i $\forall \phi \in P_2$)



Proof The lagrange nodal variables $\{N_i\}_1^6$ (misprint in the book) determine P_2. Let $\{\phi_i\}$ be the dual nodal basis: $N_i(\phi_j) = \delta_{ij}$. Let $\phi \in P_2$.

Then $\phi = \sum_{i=1}^{6} N_i(\phi)\phi_i = \sum_1^6 \phi(z_i)\phi_i$

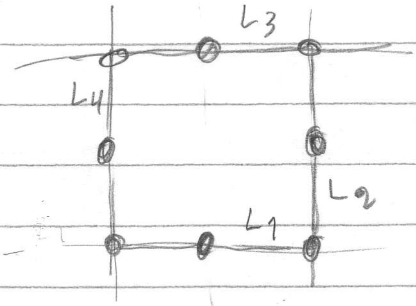
and $\phi(z_9) = \sum_1^6 \phi(z_i)\underbrace{\phi_i(z_9)}_{=c_i}$

Take $c_i = \begin{cases} \phi_i(z_9), & i = 1, \dots, 6 \\ 0, & i = 7, 8. \end{cases}$ $\square$

Using these c_i we define

$$\mathcal{P} = \{ \phi \in Q_2 : \sum_{i=1}^{8} c_i \phi(z_i) - \phi(z_9) = 0 \}$$

$\dim(\mathcal{P}) = 8$



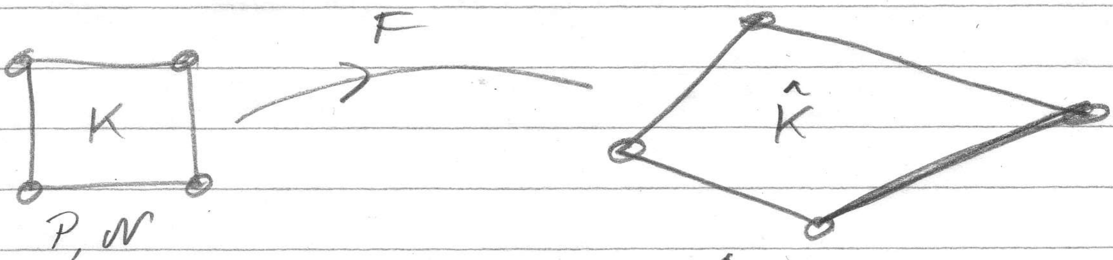
Determined by $\mathcal{N} = \{N_i\}_1^8$

because $N_i(\phi) = 0$

$$\Rightarrow \quad \phi = c\, L_1 L_2 L_3 L_4$$

But $0 = \phi(z_9) = c\, L_1(z_9) L_2(z_9) L_3(z_9) L_4(z_9) \Rightarrow c = 0.$

Non-rectangular quadrilateral elements can be obtained by affine transformation:



$\mathcal{P}, \mathcal{N}$

$\hat{K} = F(K)$

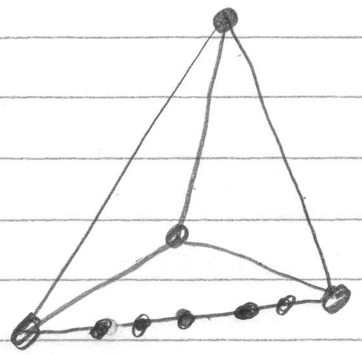
$\hat{\mathcal{P}} = (F^{-1})^*(\mathcal{P})$

$\hat{\mathcal{N}} = F_*(\mathcal{N})$

3.6 Higher dimension

3-D simplicial element

tetrahedron



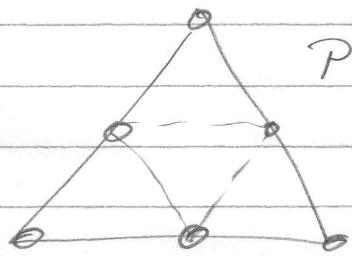
vertices 4

interior edge nodes $(k-1) * 6$

interior face nodes $\frac{(k-2)(k-1)}{2} * 4$ (if $k \geq 3$)

interior nodes : $\dim P_{k-4}$ (if $k \geq 4$)

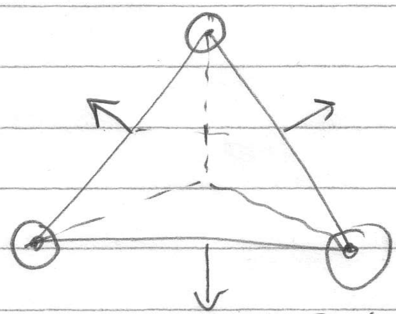
3.7 exotic elements



P p.w. linear in K

C^0

macro-p.w. linear



C^1

Clough-Tocher

macro-p.w. cubic