Lecture 6 Jan 30, 2012

Chapter 1 Review of Sobolev spaces

1.1 Lebesgue spaces

Usually in PDE theory:
domain Ω = open, connected, $\subset \mathbb{R}^n$

But here:
domain Ω = Lebesgue measurable set $\subset \mathbb{R}^n$.

Recall: $L_p(\Omega)$

$$\|f\|_{L_p(\Omega)} = \begin{cases} \left(\int_\Omega |f|^p dx\right)^{\frac{1}{p}}, & p \in [1, \infty) \\ \operatorname{ess\,sup}_\Omega |f|, & p = \infty \end{cases}$$

Banach spaces.

Hölder: $\|fg\|_{L_1} \le \|f\|_{L_p} \|g\|_{L_{p'}}$, $\frac{1}{p} + \frac{1}{p'} = 1$

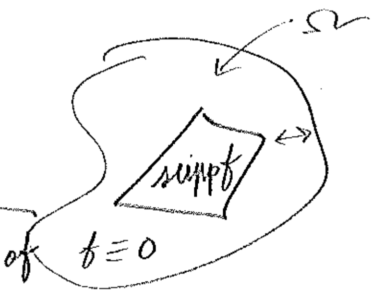
$f = g$ in L_p $\Leftrightarrow$ $f(x) = g(x)$ for almost every x

1.2 Weak derivative

Test functions: $f \in \mathcal{D}(\Omega) = C_0^\infty(\Omega)$

$\operatorname{supp} f = \overline{\{x : f(x) \neq 0\}}$ compact support in Ω

the interior of



$L_{1,loc}(\Omega) = \{f : f \in L^1(K) \ \forall \text{ compact } K \subset \operatorname{int}(\Omega)\}$

(ignores the behavior near $\partial\Omega$)

Distribution derivative of $f \in L_{1,loc}(\Omega)$:

$$D_d^\alpha f(\phi) := (-1)^{|\alpha|} \int_\Omega f(x)\, D^\alpha \phi(x)\, dx \quad \forall \phi \in \mathcal{D}(\Omega)$$

always defined but may be very "bad"
(Dirac delta for example)

$f \in L_{1,loc}(\Omega)$ has a <u>weak derivative</u> $D_w^\alpha f$

if $\exists\, g \in L_{1,loc}(\Omega)$:

$$\int_\Omega g\,\phi\,dx = D_d^\alpha f(\phi) = (-1)^{|\alpha|} \int_\Omega f\, D^\alpha \phi\, dx$$

Then $D_w^\alpha f := g$..

<u>Clearly: $D_w^\alpha f = D^\alpha f$ if $f \in C^{|\alpha|}(\Omega)$.</u>

Note: $g \in L_{1,loc}$ can be more "bad"

in higher dimension, e.g.,

$$g(x) = |x|^\gamma$$

$$\int_B |g|\,dx = \int_0^1 r^\gamma \cdot r^{n-1}\,dr < \infty \iff \gamma + n - 1 > -1$$

$$\iff \gamma > -n$$

But, still, not very bad.

1.3 <u>Sobolev spaces</u> ($k \geq 0$ integer)

$$\|f\|_{W_p^k(\Omega)} = \begin{cases} \left(\sum_{|\alpha| \le k} \|D^\alpha f\|_{L_p(\Omega)}^p \right)^{1/p}, & p \in [1, \infty) \\ \max_{|\alpha| \le k} \|D^\alpha f\|_{L_\infty(\Omega)}, & p = \infty. \end{cases}$$

$$W_p^k(\Omega) := \{ f \in L_{1,loc}(\Omega) : \|f\|_{W_p^k(\Omega)} < \infty \}$$

Banach space. Seminorms: $|f|_{W_p^k}$ $|\alpha| = k$

Note: $W_\infty^1(\Omega) = Lip(\Omega)$ with eq norms. (some assumpt. Ω)

$$\|f\|_{Lip(\Omega)} = \|f\|_{L_\infty(\Omega)} + \sup_{x,y \in \Omega} \left\{ \frac{|f(x) - f(y)|}{|x-y|} \right\}$$

$n=1$ $W_1^1(\Omega)$ = absol. cont^s

Another type of Sobolev space:

$$H_p^k(\Omega) = \overline{C^k(\Omega)}^{\|\cdot\|_{W_p^k}}, \quad p \in [1, \infty)$$

Note: $H_\infty^k(\Omega) = C^k(\Omega) \neq W_\infty^k(\Omega) = Lip_k(\Omega)$.

But Meyers + Serrin 1964 : $H = W$

<u>Theorem</u> Ω open. Then $C^\infty(\Omega) \cap W_p^k(\Omega)$ is dense in $W_p^k(\Omega)$ for $p < \infty$.

Another type of density result:

$$C^\infty(\bar{\Omega}) \text{ dense in } W_p^k(\Omega)$$

if $\partial\Omega$ is not too bad.

no cusp!

1.4 <u>Inclusions</u>

$$\begin{cases} W_p^m(\Omega) \subset W_p^k(\Omega) & \text{if } m \geq k \\ \|f\|_{W_p^k(\Omega)} \leq \|f\|_{W_p^m(\Omega)} & \end{cases}$$

$$\begin{cases} W_q^k(\Omega) \subset W_p^k(\Omega) & \text{if } \Omega \ \underline{\text{bdd}}, \ q \geq p \\ \|f\|_{W_p^k(\Omega)} \leq C\|f\|_{W_q^k(\Omega)} & \left(\text{Hölder}, \ C = |\Omega|^{\frac{1}{p} - \frac{1}{q}}\right) \end{cases}$$

Assume Ω has Lipschitz boundary $\partial\Omega$:

essentially, $\partial\Omega$ is locally the graph of a Lip fcn with some uniformity.

No cusp: 

Then $\exists$ extension operator:

$$E : W_p^k(\Omega) \longrightarrow W_p^k(\mathbb{R}^n)$$

$$Ef|_\Omega = f$$

$$\|Ef\|_{W_p^k(\mathbb{R}^n)} \le C \|f\|_{W_p^k(\Omega)}$$

Sobolev's ineq. (Proved in Chapt 4)

If $k - \frac{n}{p} > 0$ $\quad p \in (1, \infty)$

$k - n \geq 0$ $\quad p = 1$ $\quad$ ($k = n$ means 1 deriv in each direction)

then $\begin{cases} W_p^k(\Omega) \subset C(\Omega) \ \left(\text{or } C(\bar{\Omega}) \text{ if } \partial\Omega \text{ is not too bad } \underset{4, X.20}{}\right) \\ \|f\|_{L_\infty(\Omega)} \le C \|f\|_{W_p^k(\Omega)} \end{cases}$

If $k - \frac{n}{p} > m$ $(p > 1)$, $\quad k - n \geq m$ $(p = 1)$ then

$\begin{cases} W_p^k(\Omega) \subset C^m(\bar{\Omega}) \\ \|f\|_{W_\infty^m} \le C \|f\|_{W_p^k} \end{cases}$

Also:

$$k - \frac{n}{p} \geq m - \frac{n}{q} \quad \left(\text{with} > \text{ if } m - \frac{n}{q} \text{ integer}\right)$$

$$\Rightarrow \begin{cases} W_p^k(\Omega) \subset W_q^m(\Omega) \\ \|f\|_{W_q^m} \leq C \|f\|_{W_p^k} \end{cases}$$

1.6 Trace theorems

Assume Lip bdry, $1 \leq p \leq \infty$.

Then $\exists\, \gamma : W_p^1(\Omega) \to L_p(\partial\Omega)$, $\gamma v = v|_{\partial\Omega}$ if $v \in C(\bar\Omega)$ and

$$\|\gamma v\|_{L_p(\partial\Omega)} \leq C \|v\|_{L_p(\Omega)}^{1-\frac{1}{p}} \|v\|_{W_p^1(\Omega)}^{\frac{1}{p}}$$

Then we may define

$$\overset{\circ}{W}{}_p^1(\Omega) = \{ v \in W_p^1 : \gamma v = 0 \}$$

1.7 Negative order norms

B Banach, B' dual space, $\|L\|_{B'} = \sup_{f \in B} \frac{|L(f)|}{\|f\|_B}$

$$(L_p(\Omega))' = L_{p'}(\Omega), \quad \frac{1}{p} + \frac{1}{p'} = 1 \quad 1 \leq p < \infty$$

$$W_p^{-k}(\Omega) = (W_{p'}^k(\Omega))', \quad k \geq 0$$

$$\|L\|_{W_p^{-k}} = \sup \frac{|L(f)|}{\|f\|_{W_{p'}^k}}$$

Dirac $\delta_{x_0}(f) = f(x_0)$

$|\delta_{x_0}(f)| = |f(x_0)| \leq \|f\|_{L_\infty} \leq$

$\leq C \|f\|_{W_{p'}^k} \quad k - \frac{n}{p'} > 0$

$\delta_{x_0} \in W_p^{-k}(\Omega)$.