# Lecture 8 Feb 3

## 4.3 Riesz potentials

**4.3.1 Lemma** Let $f \in L_p(\Omega)$, $1 < p < \infty$, $m > n/p$, or $p = 1$, $m = n$.

Then
$$\int |z - x|^{-n+m} |f(z)| \, dz \le C_p \, d^{m - n/p} \, \|f\|_{L_p(\Omega)}$$

**Proof** First $1 < p < \infty$, $m > n/p$.

$$\int |x - z|^{-n+m} |f(z)| \, dz \overset{\text{Hölder}}{\le} \left( \int_\Omega |x - z|^{(-n+m)q} dz \right)^{\frac{1}{q}} \|f\|_{L_p}$$

$$\le c \left( \int_0^d r^{(-n+m)q + n - 1} dr \right)^{1/q} \|f\|_{L_p}$$

$$= c \left( \frac{d^{(-n+m)q+n}}{(-n+m)q+n} \right)^{\frac{1}{q}} \|f\|_{L_p}$$

$$= C_{n,p,m} \, d^{m - n\overbrace{\left(1 - \frac{1}{q}\right)}^{= 1/p}} \|f\|_{L_p}$$

$p = 1$, $m = n$: $\quad$ —"— $\le \|(x - \cdot)^{-n+m}\|_{L_\infty} \|f\|_{L_1} = d^{m-n} \|f\|_{L_1}$.

(Note: $p \to \infty \Rightarrow q \to 1$ so $c$ is bdd as $p \to \infty$)

**4.3.2 Prop** $m,p$ as before, $u \in W_p^m(\Omega) \Rightarrow$

$$\|R^m u\|_{L_\infty(\Omega)} \le C_{m,n,\gamma,p}\, d^{\,m-n/p}\, |u|_{W_p^m(\Omega)}$$

**Proof** Let $u \in C^m(\Omega) \cap W_p^m(\Omega)$. $\overset{(4.2.8)}{\underset{(4.2.18)}{\Longrightarrow}}$

$$|(R^m u)(x)| = m \Big| \sum_{|\alpha|=m} \int_{C_x} \overbrace{k_\alpha(x,z)}^{= (x-z)^\alpha k(x,z)} D^\alpha u(z)\, dz \Big|$$

$$\le C_{m,n,\gamma} \sum_{|\alpha|=m} \int_\Omega |x-z|^{m-n}\, |D^\alpha u(z)|\, dz$$

$$\overset{(4.3.1)}{\le} C_{m,n,\gamma,p}\, d^{\,m-n/p}\, |u|_{W_p^m(\Omega)}$$

Density argument completes the proof (Problem 4.X.18). □

**4.3.4 Sobolev's ineq.**

Assume $\operatorname{diam} \Omega = d$, star-shaped w.r.t $B$, $u \in W_p^m(\Omega)$, $1<p<\infty$, $m-n/p>0$ or $p=1$, $m-n=0$. Then $u \in C^\infty(\Omega)$ and

$$\|u\|_{L_\infty(\Omega)} \le C_{m,n,\gamma,d}\, \|u\|_{W_p^m(\Omega)}.$$

## Proof

$$\|u\|_{L_\infty(\Omega)} \le \|u - Q^m u\|_{L_\infty} + \|Q^m u\|_{L_\infty} \le$$

$$(*) \qquad \le C_{m,n,\gamma} |u|_{W^m_p(\Omega)} + C_{m,n,\gamma} \|u\|_{L_1(\Omega)}$$

$$\le C \|u\|_{W^m_p(\Omega)} .$$

Serrin + Meyers: $C^\infty(\Omega) \cap W^m_p(\Omega)$ denser in $W^m_p(\Omega)$.

Take $\{u_j\} \subset C^\infty \cap W^m$, $u_j \to u$ in $W^m_p$.

Then $(*) \Rightarrow \|u - u_j\|_{L_\infty} \le C \|u - u_j\|_{W^m_p} \to 0$

$\Rightarrow u_j \to u$ uniformly

$\Rightarrow u$ cont$^s$. $\square$

Recall 4.3.2

$$\|R^m u\|_{L_\infty} \le C_{m,n,\gamma,p}\, d^{m-n/p} |u|_{W^m_p}$$

$$p \to \infty \qquad \|R^m u\|_{L_\infty} \le C\, d^m |u|_{W^m_\infty}$$

Now we want to put a p here! (Bramble-Hilbert)

4.3.6 Lemma $\quad 1 \le p \le \infty, \quad m \geq 1.$

If $\quad g(x) = \int_\Omega |x-z|^{m-n} |f(z)|\, dz$

then

$$\|g\|_{L_p(\Omega)} \le C_{m,n}\, d^m\, \|f\|_{L_p(\Omega)}.$$

(Note: Lemma 4.3.1 has $p=\infty$ on the left.)

Proof First $1 < p < \infty$.

$$\|g\|^p_{L_p} = \int_\Omega \Big| \int_\Omega \underbrace{|x-z|^{\overbrace{(m-n)}^{1=\frac1p+\frac1q}} |f(z)|}_{|x-z|^{\frac1p(m-n)}|f(z)| \cdot |x-z|^{\frac1q(m-n)}}\, dz \Big|^p dx$$

Hölder

$$\le \int_\Omega \Big\{ \Big(\int_\Omega |z-x|^{m-n} |f(z)|^p dz\Big)^{\frac1p} \Big(\underbrace{\int_\Omega |x-z|^{m-n} dz}_{\substack{= c\int_0^d r^{m-n+n-1} dr \\ = cd^m}}\Big)^{\frac1q} \Big\}^p dx$$

$$= c\, d^{mp/q} \int \int |z-x|^{m-n} |f(z)|^p dz\, dx$$

Fubini

$$= c\, d^{mp/q} \underbrace{\int |z-x|^{m-n} dx}_{=cd^m}\ \|f\|^p_{L_p}$$

$$= c\, d^{mp\underbrace{\left(\frac1q+\frac1p\right)}_{=1}} \|f\|^p_{L_p}$$

$p=1$: $\|g\|_{L_1} = \int \int |x-z|^{m-n} |f(z)|\, dz\, dx$

Fubini
$= \int \underbrace{\int |x-z|^{m-n} dx}_{=cd^m} |f(z)|\, dz = c\, d^m \|f\|_{L_1}$

$p=\infty$:
$$|g(x)| = \int |x-z|^{m-n} |f(z)|\, dz$$
$$\le \underbrace{\int |x-z|^{m-n} dz}_{=cd^m} \|f\|_{L_\infty}$$
$$\|g\|_{L_\infty} \le c\, d^m \|f\|_{L_\infty}.$$
$\square$

## 4.3.8 Bramble-Hilbert lemma

$$|u - Q^m u|_{W_p^k(\Omega)} \le C_{m,n,\gamma}\, d^{m-k} |u|_{W_p^m(\Omega)}.$$

for $k = 0, \ldots, m$, $u \in \overline{W}_p^m(\Omega)$.

~~May assume d=1, by scaling argument.~~ Not needed!

Proof Choose $B = B_\rho$ so that $\rho \ge \frac{1}{2}\rho_{max}$

hence $\cancel{\gamma}\, \frac{d}{\rho} \le 2\gamma$.

~~Proof~~ ~~May assume d~~

$k=m$: $|u - Q^m u|_{W_p^m} = |u|_{W_p^m}$, because $D^\alpha Q^m u = 0$ if $|\alpha| = m$

$k=0$: $\|u - Q^m u\|_{L_p} = \|R^m u\|_{L_p} \le$

$$\le m \sum_{|\alpha|=m} \left\| \int_\Omega k_\alpha(x,z) D^\alpha u(z)\, dz \right\|_{L_p}$$

$$\overset{4.2.18}{\le} C_{m,n} (1+\gamma)^n \sum_{|\alpha|=m} \underbrace{\left\| \int |z-x|^{m-n} |D^\alpha u(z)|\, dz \right\|_{L_p}}_{\le C d^m \|D^\alpha u\|_{L_p}}$$

$$= C d^m |u|_{W_p^m(\Omega)}$$

$0 < k < m$:

$$|u - Q^m u|_{W_p^k(\Omega)} = |R^m u|_{W_p^k} = \sum_{|\alpha|=k} \| \underbrace{D^\alpha R^m u}_{(4.1.18)\ = R^{m-k} D^\alpha u} \|_{L_p}$$

$$= \sum \|R^{m-k} D^\alpha u\|_{L_p}$$

previous case $\rightarrow$ $\le C d^{m-k} \sum_\alpha \|D^\alpha u\|_{L_p} = C d^{m-k} |u|_{W_p^m}$

## (4.3.14) Lemma Friedrichs' ineq

$\Omega$ star-shaped $\Rightarrow$

$$\|u-\bar{u}\|_{W_p^1(\Omega)} \le C_{m,\gamma}\, |u|_{W_p^1(\Omega)}$$

with $\bar{u} = \frac{1}{|\Omega|}\int u\,dx$.

<u>Proof</u> Let $c$ be arbitrary.

$$\|u-\bar{u}\|_{L_p} = \|u-c+\overline{u-c}\|_{L_p} \le \|u-c\|_{L_p} + \underbrace{\|\overline{u-c}\|_{L_p}}_{\le \|u-c\|_{L_p}}$$

$$\le 2\|u-c\|_{L_p}$$

so that

$$\|u-\bar{u}\|_{L_p} \le 2\inf_c \|u-c\|_{L_p} \overset{c=Q^1u}{\le} 2\|u-Q^1u\|_{L_p}$$

$$\le C\,|u|_{W_p^1(\Omega)}$$

Finally:

$$\|u-\bar{u}\|_{W_p^1}^p = \underbrace{\|u-\bar{u}\|_{L_p}}_{\le C|u|_{W_p^1}} + \underbrace{|u-\bar{u}|_{W_p^1}}_{=|u|_{W_p^1}}$$

$$\le C\,|u|_{W_p^1}.$$

$\square$