# Lecture 9 Feb. 20

## 4.4 Bounds for the interpolator.

Recall:

Finite element $(K, P, \mathcal{N})$

$$\begin{cases} K \subset \mathbb{R}^n \\ P \text{ shape functions, basis } \{\phi_i\}_{i=1}^k \\ \mathcal{N} = \{N_i\}_{i=1}^k \text{ nodal variables} \end{cases}$$

Bramble-Hilbert lemma

$\Omega$ starshaped with respect to $B = B_\rho$, $\rho \geq \frac{1}{2}\rho_{max}$. Then, with $d = \text{diam}(\Omega)$, $k = 0, \dots, m$,

$$|u - Q_B^m u|_{W_p^k(\Omega)} \leq C_{m,n,\gamma}\, d^{m-k}\, |u|_{W_p^m(\Omega)}$$

Chunkiness: $\gamma = \dfrac{d}{\rho_{max}} \geq 2$ ($\gamma \gg 2$ thin domain)

$C_{m,n,\gamma} \to \infty$ as $\gamma \to \infty$

Wrong: not need

**4.4.1 Lemma**

$(K, \mathcal{P}, \mathcal{N})$ with $\mathrm{diam}(K) = 1$

$\mathcal{P} \subset W^m_\infty(K)$ ($K$ is closed)

$\mathcal{N} \subset (C^\ell(K))'$

Then $I : C^\ell(K) \to W^m_p(K)$, is bdd for $1 \le p \le \infty$.

**Proof** $\quad I u = \sum_{i=1}^{k} N_i(u) \phi_i$

$$\| I u \|_{W^m_p(K)} \le \sum_{i=1}^{k} \underbrace{|N_i(u)|}_{\le \|N_i\|_{(C^\ell(K))'} \|u\|_{C^\ell(K)}} \|\phi_i\|_{W^m_p(K)}$$

$$\le \left( \sum_{i=1}^{k} \|N_i\|_{C^\ell(K)'} \|\phi\|_{W^m_p(K)} \right) \|u\|_{C^\ell(K)}$$

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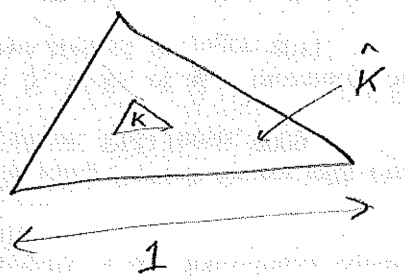
**4.4.2 Def.** $\quad \sigma_p(K) = \|I\|_{C^\ell(K) \to W^m_p(K)}$

**4.4.3 Def.** $\quad \hat{K} = \{ \mathrm{diam}(K)^{-1} x : x \in K \}$

dilation: same shape



## 4.4.4 Theorem (Local interpolation error) (3

$(K, P, \mathcal{N})$

(i) $K$ is star-shaped w.r.t. some ball and with ch. $\gamma$.

(ii) $P_{m-1} \subset P \subset W^m_\infty(K)$

(iii) $\mathcal{N} \subset (C^l(K))'$

$m - \frac{n}{p} > l$, $p > 1$

$m - \frac{n}{p} \geq l$, $p = 1$

(Sobolev: $\|v\|_{C^l(K)} \leq C_{m,n,\gamma,d} \|v\|_{W^m_p(K)}$ so that $I$ is defined)

Then

$$|v - Iv|_{W^i_p(K)} \leq C_{m,n,\gamma,\sigma(\hat{K})} \operatorname{diam}(K)^{m-i} |v|_{W^m_p(K)},$$

$$i = 0, \dots, m.$$

Proof 1) Prove it for $\hat{K}$. Let $Q^m$ be as in Bramble-Hilbert for $\hat{K}$. $\hat{K}$ has the same $\gamma$.

Note: $IQ^m v = Q^m v$ because $Q^m v \in P_{m-1} \subset P$.

$$|v - Iv|_{W^i_p(\hat{K})} \leq |v - Q^m v|_{i,p} + |Q^m v - Iv|_{i,p}$$

$$= |v - Q^m v|_{i,p} + |I(Q^m v - v)|_{i,p}$$

$$\leq |v - Q^m v|_{W^i_p(\hat{K})} + \sigma(\hat{K}) \|Q^m v - v\|_{C^l(\hat{K})}$$

$(d=1)$

$$\overset{\text{Sobolev}}{\leq} \left(1 + \sigma(\hat{K}) C_{m,n,\gamma,1}\right) \|v - Q^m v\|_{W^m_p(\hat{K})}$$

$$\overset{\text{Br.-H.}}{\leq} C_{m,n,\gamma,\sigma(\hat{K})} |v|_{W^m_p(\hat{K})}$$

2) Homogeneity argument. (4.X.5)

$$|v|_{W_p^i(\hat{K})} = \mathrm{diam}(K)^{i-n/p}\,|v|_{W_p^i(K)}$$

semi-norm !! $\qquad \hat{x} = d^{-1}x$

$\qquad \hat{v}(\hat{x}) = v(d\hat{x})$

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## 4.4.7 Corollary

$$|v - Iv|_{W_\infty^i(K)} \le C_{m,n,\gamma,\sigma(\hat{K})}\,\mathrm{diam}(K)^{m-i-n/p}\,|v|_{W_p^m(K)}$$

for $i = 0, \dots, l$.

Proof $\quad |v - Iv|_{W_\infty^i(\hat{K})} \overset{\text{Sobolev}}{\le} C_{m,n,\gamma}\,\|v - Iv\|_{W_p^m(\hat{K})}$

$\qquad \overset{4.4.6}{\le} C_{m,n,\gamma,\sigma(\hat{K})}\,|v|_{W_p^m(\hat{K})}$

$$|v - Iv|_{W_\infty^i(K)} = d^{-i+\frac{n}{\infty}}\,|v - Iv|_{W_\infty^i(\hat{K})} \le C d^{-i}\,|v|_{W_p^m(\hat{K})} = C d^{-i+m-n/p}\,|v|_{W_p^m(K)}$$

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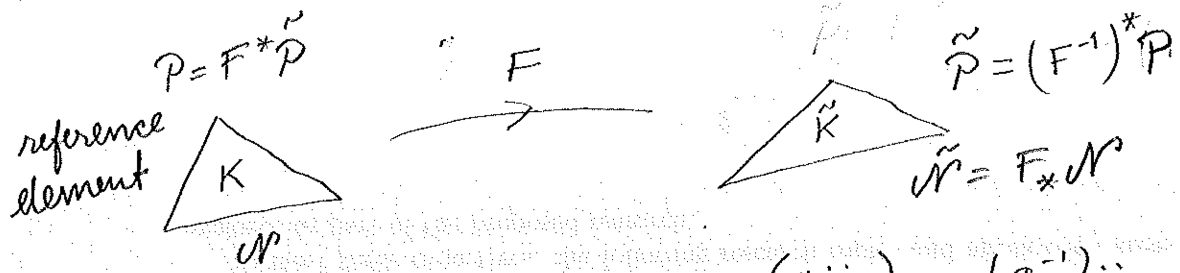
$C_{m,n,\gamma,\sigma(\hat{K})}$

How does $\sigma(\hat{K})$ change under affine transformation?




$$\tilde{x} = F(x) = ax + b \qquad a = (a_{ij}), \quad (a^{-1})_{ij}$$

$$F^*(\tilde{f}) = \tilde{f} \circ F, \qquad \tilde{N} = F_*(N) \ \forall N \in \mathcal{N}$$

$$\tilde{N}(\tilde{f}) = (F_* N)(\tilde{f}) = N(F^*(\tilde{f})) = N(\tilde{f} \circ F) \quad \forall N \in \mathcal{N}$$

Fix $(K, P, \mathcal{N})$ and compute $\sigma(\tilde{K})$ as a function of $F$.

$$\| \tilde{I} \tilde{v} \|_{W^m_p(\tilde{K})} = \| \sum_{\tilde{N} \in \tilde{\mathcal{N}}} \tilde{N}(\tilde{v}) \, \tilde{\phi}_{\tilde{N}} \| = \| \sum_{N \in \mathcal{N}} (F_*(N))(\tilde{v}) (F^{-1})^*(\phi_N) \|_{W^m_p(\tilde{K})}$$

$$= \| \sum_N N(\tilde{v} \circ F) \, \phi_N \circ F^{-1} \|_{W^m_p(\tilde{K})}$$

$$\leq \sum_N | N(\tilde{v} \circ F) | \, \| \phi_N \circ F^{-1} \|_{W^m_p(\tilde{K})}$$

$$\leq \sum_N C_N \| \tilde{v} \circ F \|_{C^\ell(K)} \| \phi_N \circ F^{-1} \|_{W^m_p(\tilde{K})}$$

$$\leq \sum_N C_N \, C_{n,\ell} \left(1 + \max_{i,j} |a_{ij}|\right)^\ell \| \tilde{v} \|_{C^\ell(\tilde{K})} \cdot C_{n,m} \left(1 + \max |a^{-1}_{ij}|\right)^m |\det(a)|^{1/p} \times \| \phi_N \|_{W^m_p(K)}$$

$$\leq C_{(K,P,\mathcal{N})} \left(1 + \max |a_{ij}|\right)^\ell \left(1 + \max |a^{-1}_{ij}|\right)^m |\det(a)|^{1/p} \| \tilde{v} \|_{C^\ell(\tilde{K})}$$

$$\sigma(\tilde{K}) \leq C_{(K,P,\mathcal{N})} \underbrace{\left(1 + \max |a_{ij}|\right)^\ell \left(1 + \max |a^{-1}_{ij}|\right)^m |\det(a)|^{1/p}}_{=: \chi(a)}$$

e.g. $\displaystyle \frac{\partial}{\partial x_j} \tilde{v}(ax+b) = \sum_i \frac{\partial \tilde{v}}{\partial \tilde{x}_i}(ax+b) \frac{\partial \tilde{x}_i}{\partial x_j} = \sum_i a_{ij} \frac{\partial \tilde{v}}{\partial \tilde{x}_i}$

$$\| \tilde{v} \circ F \|_{C^1(K)} = \max |\tilde{v} \circ F| + \max \left| \frac{\partial \tilde{v} \circ F}{\partial x_j} \right| \leq C (1 + \max |a_{ij}|) \| \tilde{v} \|_{C^1}$$

## 4.4.11 Proposition

$$\sigma(\tilde{K}) \leq C_{ref}\, \chi(a)$$

for all elements $(\tilde{K}, \tilde{P}, \tilde{N})$ that are affine equivalent to a fixed reference element $(K, P, N)$ with affine mapping $F(x) = ax + b$. Here $C_{ref}$ depends on the ref. element and $\chi \in C(GL(\mathbb{R}^n))$.

$GL(\mathbb{R}^n)$ = general linear group on $\mathbb{R}^n$ =
= non-singular $n \times n$ matrices.