1. Weighted norm error estimates

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1.1. Notation. Define a mesh in the interval $I = (0,1)$:

$$
\begin{aligned}
&0 = x_0 < x_1 < \cdots < x_{j-1} < x_j < \cdots < x_n = 1, \\
&h_j = x_j - x_{j-1}, \quad I_j = (x_{j-1}, x_j), \quad j = 1, \dots, n.
\end{aligned} \tag{1.1}
$$

We define the piecewise constant mesh size function $\bar{h}$ by

$$
\bar{h}(x) = h_j, \quad x \in I_j, \tag{1.2}
$$

and a piecewise linear mesh size function h with node values

$$
h(x_j) = h_j + h_{j+1}, \ j = 0, \dots, n, \quad \text{with } h_0 = h_1, \ h_{n+1} = h_n. \tag{1.3}
$$

It follows that

$$
h(x) \geq h_j = \bar{h}(x), \quad x \in I_j. \tag{1.4}
$$

It will be important to know that h is not much larger than the true mesh size $\bar{h}$. Then we must put some restriction on the variation of the mesh. We define the mesh ratios

$$
r_j = \frac{h_{j+1}}{h_j}, \quad j = 1, \dots, n-1. \tag{1.5}
$$

We assume that there is $\gamma \geq 1$ such that

$$
r_j \in [\gamma^{-1}, \gamma], \quad j = 1, \dots, n-1. \tag{1.6}
$$

Of course, this is trivially true for a single mesh, but it is important to understand that we assume that this holds for all meshes in the mesh family that we consider. This could be a finite family of a single mesh or a few meshes, but usually an infinite family of successively refined meshes with $h_{\max} \to 0$. We may also think of γ as a parameter which gives a quantitative measure of the quality of the mesh family (for a finite or an infinite mesh family). This is a rather weak restriction. It allows for example a geometrically graded mesh family: $h_{j+1} = \gamma h_j = \cdots = \gamma^j h_0$, where $h_0 \to 0$, so that $h_{\max} = h_n = \gamma^{n-1} h_0 \to 0$. If $\gamma = 1$, then the family is uniform, i.e., $h_j = h_{\max}$ and $h = 2\bar{h}$.

In general we have

$$
\begin{aligned}
h(x) &= (h_j + h_{j+1})\phi_{j-1}(x) + (h_{j-1} + h_j)\phi_j(x) \\
&= (h_j + h_{j+1}) \frac{x - x_{j-1}}{h_j} + (h_{j-1} + h_j) \frac{x_j - x}{h_j} \\
&= h_j \Big((1 + r_j) \frac{x - x_{j-1}}{h_j} + (r_{j-1}^{-1} + 1) \frac{x_j - x}{h_j} \Big), \quad x \in I_j,
\end{aligned} \tag{1.7}
$$

and it follows that

$$
\bar{h}(x) \leq h(x) \leq (1 + a)\bar{h}(x). \tag{1.8}
$$

Similarly

$$
h'(x) = r_j - r_{j-1}^{-1}, \quad x \in I_j, \tag{1.9}
$$

so that

$$
\|h'\|_{L_\infty} \leq a - a^{-1} \to 0 \quad \text{as } a \to 1^+. \tag{1.10}
$$

1.2. **An interpolation error estimate.** Let

$$S = \{v \in C([0,1]) : v|_{I_j} \in \Pi_1,\ v(0) = v(1) = 0\}$$

and the interpolator $I_S \colon C([0,1]) \to S$ with

$$v_I(x) = (I_S v)(x) = \sum_{j=1}^{n-1} v(x_j)\phi_j(x).$$

We introduce the L_2-norm and the piecewise L_2-norm:

$$\begin{aligned} \|v\| &= \|v\|_{L_2(I)} = \Big(\int_0^1 v^2\,dx\Big)^{1/2}, \\ \|v\|_{\mathrm{PW}} &= \Big(\sum_{j=1}^{n} \|v\|^2_{L_2(I_j)}\Big)^{1/2}. \end{aligned} \tag{1.11}$$

We recall the local interpolation error estimate, see the proof of (0.4.5) in Brenner-Scott, for $v \in H^2$ with $v(0) = v(1) = 0$,

$$\|(v - v_I)'\|^2_{L_2(I_j)} \le \tfrac{1}{2} h_j^2 \|v''\|^2_{L_2(I_j)}. \tag{1.12}$$

Toghether with (1.4) this immediately implies the following two estimates in global weighted norms

$$\|(v - v_I)'\| \le \tfrac{1}{\sqrt{2}} \|hv''\|_{\mathrm{PW}}, \tag{1.13}$$

$$\|h^{-1}(v - v_I)'\| \le \tfrac{1}{\sqrt{2}} \|v''\|_{\mathrm{PW}}. \tag{1.14}$$

Of course these hold equally well with h replaced by $\bar{h}$.

1.3. **Energy norm error estimate.** Consider the BVP:

$$-(au')' = f \quad \text{in } I; \qquad u(0) = u(1) = 0.$$

Let $V = H_0^1 = H_0^1(I)$. The weak formulation is

$$u \in V; \quad (au', v') = (f, v) \quad \forall v \in V, \tag{1.15}$$

and the FEM

$$u_S \in S; \quad (u_S', v') = (f, v) \quad \forall v \in S. \tag{1.16}$$

Then, since $S \subset V$, we have the Galerkin orthogonality

$$(a(u_S - u)', v') = 0 \quad \forall v \in S. \tag{1.17}$$

and, by (1.13),

$$\|\sqrt{a}(u_S - u)'\| = \inf_{v \in S} \|\sqrt{a}(v - u)'\| \le \|a\|^{1/2}_{L_\infty} \|(u_I - u)'\| \le \tfrac{1}{\sqrt{2}} \|a\|^{1/2}_{L_\infty} \|hu''\|, \tag{1.18}$$

if $f \in L_2$ so that $u'' \in L_2$. Since $\sqrt{a_0}\|(u_S - u)'\| \le \|\sqrt{a}(u_S - u)'\|$, we conclude

$$\|(u_S - u)'\| \le C\|hu''\| = C\Big(\sum_{j=1}^{n} h_j^2 \|u''\|^2_{L_2(I_j)}\Big)^{1/2} \tag{1.19}$$

$$\le C h_{\max} \|u''\|, \tag{1.20}$$

with $C = (\|a\|_{L_\infty}/(2a_0))^{1/2}$. This also holds with h replaced by $\bar{h}$. Note that the weighted expression on the right of (1.19) contains more information than (1.20). We want to prove a similar weighted error estimate for the L_2-norm of the error.

1.4. L_2**-norm errror estimate.** We argue by duality. Let $e = u_S - u$. We use the dual problem

$$w \in V; \quad (av', w') = (v, e) \quad \forall v \in V. \tag{1.21}$$

Then, by taking $v = e$ and using (1.17), (1.14) and $\|w''\| \le C\|e\|$,

$$\begin{aligned} \|e\|^2 = (ae', w') = (e', w' - w_I') &\le \|\sqrt{a}he'\| \, \|\sqrt{a}h^{-1}(w - w_I)'\| \\ &\le C\|he'\| \, \|h^{-1}(w - w_I)'\| \le C\|he'\| \tfrac{1}{\sqrt{2}}\|w''\| \le C\|he'\| \|e\|. \end{aligned} \tag{1.22}$$

Hence

$$\|e\| \le \tfrac{1}{\sqrt{2}}\|he'\|. \tag{1.23}$$

If this calculation is done with h replaced by $\bar{h}$ then with (1.20) we obtain the standard (non-weighted) estimate

$$\|u_S - u\| \le Ch_{\max}^2\|u''\|. \tag{1.24}$$

Note: no restriction on the mesh so far.

We shall prove that, for any $\epsilon > 0$ there is $\gamma \ge 1$ such that

$$\|he'\| \le C\|h^2u''\| + \epsilon\|e\|, \tag{1.25}$$

that is, for $\|h'\|_{L_\infty}$ is sufficiently small (see (1.10)), With (1.23) this leads to the weighted estimate

$$\|u_S - u\| \le C\|h^2u''\|. \tag{1.26}$$

In view of (1.8) this also holds with h replaced by $\bar{h}$ (but with a larger constant C).

To prove (1.25) we first note that

$$\|he'\|^2 \le C(ahe', he') = C(ae', h^2e') = (ae', (h^2e)') - (2ahh'e', e). \tag{1.27}$$

Note: $h \in H^1$ but $\bar{h} \notin H^1$, so we cannot use $\bar{h}$ here. Recalling (1.10), that is,

$$\|h'\|_{L_\infty} \le M := \gamma - \gamma^{-1}, \tag{1.28}$$

we get for the last term

$$|(2ahh'e', e)| \le CM\|he'\| \, \|e\|. \tag{1.29}$$

For the first term on the right side of (1.27) we get with $v = h^2e$, by (1.17), (1.14),

$$\begin{aligned} (ae', (h^2e)') = (ae', v') = (ae', v' - v_I') &\le C\|he'\| \|h^{-1}(v - v_I)'\| \\ &\le C\|he'\| \tfrac{1}{\sqrt{2}}\|v''\|_{\mathrm{PW}} = C\|he'\| \|(h^2e)''\|_{\mathrm{PW}}. \end{aligned} \tag{1.30}$$

On I_j we have

$$(h^2e)'' = h^2e'' + 4hh'e' + (2(h')^2 + 2hh'')e = h^2u'' + 4hh'e' + 2(h')^2e, \tag{1.31}$$

because $e'' = u_S'' - u'' = -u''$ and $h'' = 0$. Hence

$$\|(h^2e)''\|_{\mathrm{PW}} \le \|h^2u''\| + 4M\|he'\| + 2M^2\|e\|. \tag{1.32}$$

Inserting this and (1.29) into (1.27) and dividing by $\|he'\|$ we get

$$\|he'\| \le \tfrac{1}{\sqrt{2}}\|h^2u''\| + \tfrac{4}{\sqrt{2}}M\|he'\| + 2(\tfrac{1}{\sqrt{2}}M^2 + M)\|e\|, \tag{1.33}$$

which implies (1.25) if M is sufficiently small.

We now have

$$\|(u_S - u)'\| \le C\|hu''\| \le Ch_{\max}\|u''\| \quad \text{for any } \gamma \ge 1, \tag{1.34}$$

$$\|u_S - u\| \le C\|h^2u''\| \le Ch_{\max}^2\|u''\| \quad \text{if } \gamma \text{ is close to 1}. \tag{1.35}$$

1.5. **Remark.** From the Poincaré inequality

$$\|v\| \le C\|v'\|, \quad v \in H_0^1 \tag{1.36}$$

and (1.34) there follows the L_2-norm estimate

$$\|u_S - u\| \le C\|(u_S - u)'\| \le C\|hu''\| \le Ch_{\max}\|u''\| \quad \text{for any } \gamma \ge 1. \tag{1.37}$$

Why is this not sufficient? This is because the rate of convergence is not the optimal one for the L_2-norm of the error. If u has two derivatives then we expect that the error in u_S is $O(h_{\max}^2)$ and that the error in u_S' is $O(h_{\max})$ as $h_{\max} \to 0$. `/stig`