0.X.6

Let $w \in C^2$ with $w(0) = w(1) = 0$.

Rolle's theorem: $\exists \xi : \; w'(\xi) = 0$.

Then $\quad w'(y) = w'(\xi) + \int_\xi^y w''(x)\,dx = \int_\xi^y w''(x)\,dx$

$$w(z) = \int_0^z w'(y)\,dy = \int_0^z \int_\xi^y w''(x)\,dx$$

Hence $\quad |w'(y)| \le \int_0^1 |w''|\,dx$

$$|w(z)| \le \int_0^1 |w''|\,dx$$

In other words: $\quad I = (0,1), \; k = 0, 1$

$$\| w^{(k)} \|_{L_\infty(I)} \le \| w'' \|_{L_1(I)}$$

and $\quad \| w^{(k)} \|_{L_p(I)} \le \| w^{(k)} \|_{L_\infty(I)} \le$

$$\le \| w'' \|_{L_1(I)} \le \| w'' \|_{L_q(I)}$$

We conclude:

$$\begin{cases} \| w^{(k)} \|_{L_p(I)} \le \| w'' \|_{L_q(I)} \\ k = 0, 1, \quad p, q \in [1, \infty] \end{cases}$$

Scaling: $\begin{cases} x = x_{j-1} + h_j \hat{x}, & x \in I_j, \ \hat{x} \in I \\ \hat{w}(\hat{x}) = w(x_{j-1} + h_j \hat{x}) \end{cases}$

$$\| w^{(k)} \|_{L_p(I_j)} = \left(\int_{I_j} |w^{(k)}|^p dx \right)^{\frac{1}{p}} =$$

$$= \left\{ \begin{array}{l} dx = h_j \, d\hat{x} \\ \dfrac{dw}{dx} = \dfrac{d\hat{w}}{d\hat{x}} \dfrac{d\hat{x}}{dx} = h_j^{-1} \dfrac{d\hat{w}}{d\hat{x}} \end{array} \right\} =$$

$$= \left(\int_I \left(h_j^{-k} |\hat{w}^{(k)}| \right)^p h_j \, d\hat{x} \right)^{1/p}$$

$$= h_j^{-k+1/p} \underbrace{\| \hat{w}^{(k)} \|_{L_p(I)}}_{\leq \| \hat{w}'' \|_{L_q(I)} = h_j^2 h_j^{-1/q} \| w'' \|_{L_q(I_j)}}$$

$$\leq h_j^{2-k+\frac{1}{p}-\frac{1}{q}} \| w'' \|_{L_q(I_j)}$$

Apply with $w = u - u_I$ where

$u - u_I = 0$ at x_{j-1}, x_j, $u_I'' = 0$:

$$\| u^{(k)} - u_I^{(k)} \|_{L_p(I_j)} \leq h_j^{2-k+\frac{1}{p}-\frac{1}{q}} \| u'' \|_{L_q(I_j)}$$