1. The Bramble-Hilbert lemma

Prove the following slightly more general re-formulation of the Bramble-Hilbert lemma (4.3.8). Make a more precise statement about the constant $C(\Omega)$.

Theorem 1 (Bramble-Hilbert). *Let Ω be a bounded domain in $\mathbf{R}^d$, star-shaped with respect to a ball, and assume that F is a functional on $W_p^m = W_p^m(\Omega)$, such that*

$$\begin{aligned} &F(v) \geq 0 && \forall v \in W_p^m, \quad \textit{(non-negative)} \\ &F(v+w) \leq F(v) + F(w), && \forall v, w \in W_p^m, \quad \textit{(sub-additive)} \\ &F(v) \leq C\|v\|_{W_p^m}, && \forall v \in W_p^m, \quad \textit{(bounded)} \\ &F(v) = 0, && \forall v \in \mathcal{P}_{m-1}. \quad \textit{(annihilates polynomials)} \end{aligned}$$

Then there is a constant $C = C(\Omega)$ such that, with the corresponding seminorm,

$$F(v) \leq C|v|_{W_p^m}, \quad \forall v \in W_p^m.$$

Hint: use (4.3.8).

The reason why this formulation (but not the result in itself) is more general is that it is formulated in terms of a non-negative sub-additive functional and not in terms of a norm of a specific Taylor remainder. This is closer to the original work:

J. H. Bramble and S. R. Hilbert, *Estimation of linear functionals on Sobolev spaces with application to Fourier transforms and spline interpolation*, SIAM J. Numer. Anal. **7** (1970), 112–124. link

But the proof of (4.3.8) by Dupont and Scott (1980) is better.

2. Quadrature error

Consider the *barycentric quadrature rule*

$$\int_K v \,\mathrm{d}x \approx q_K(v) = |K| v(p_K), \text{ where } |K| = \text{area}(K), \quad p_K = \tfrac{1}{3}\sum_{l=1}^{3} p_{l,K}, \tag{2.1}$$

with $p_{l,K}$ and p_K the vertices and the barycenter of the triangle K. This quadrature rule is exact for affine functions, i.e.,

$$\int_K v \,\mathrm{d}x = |K| v(p_K), \quad \forall v \in \mathcal{P}_1. \tag{2.2}$$

This implies that the rule is accurate of order 2 so that

$$\Big| q_K(v) - \int_K v \,\mathrm{d}x \Big| \leq C h_K^2 |v|_{W_1^2(K)}. \tag{2.3}$$

Prove this by means of Bramble-Hilbert.

Analyze the *edge midpoint rule* in a similar way.

/stig