1. Weighted norm error estimates

1.1. **Notation.** Define a mesh in the interval $I = (0,1)$:

$$\begin{aligned} &0 = x_0 < x_1 < \cdots < x_{j-1} < x_j < \cdots < x_n = 1, \\ &h_j = x_j - x_{j-1}, \quad I_j = (x_{j-1}, x_j), \quad j = 1, \dots, n. \end{aligned} \tag{1.1}$$

We define a piecewise linear mesh size function h by

$$h(x_j) = h_j + h_{j+1},\ j = 0, \dots, n, \quad \text{with } h_0 = h_1,\ h_{n+1} = h_n. \tag{1.2}$$

It follows that

$$h(x) \geq h_j, \quad x \in I_j. \tag{1.3}$$

It is important to know that h is not too large. Then we must put some restriction on the variation of the mesh. We define the piecewise constant mesh size function $\bar{h}$ by

$$\bar{h}(x) = h_j, \quad x \in I_j, \tag{1.4}$$

and the mesh ratios

$$r_j = \frac{h_{j+1}}{h_j}, \quad j = 1, \dots, n-1. \tag{1.5}$$

We assume that there is $a \geq 1$ such that

$$r_j \in [a^{-1}, a], \quad j = 1, \dots, n-1. \tag{1.6}$$

This is a rather weak restriction. It allows for example a geometric grading of the mesh: $h_{j+1} = a h_j$. Then

$$\begin{aligned} h(x) &= (h_j + h_{j+1}) \frac{x - x_{j-1}}{h_j} + (h_{j-1} + h_j) \frac{x_j - x}{h_j} \\ &= h_j \Big((1 + r_j) \frac{x - x_{j-1}}{h_j} + (r_{j-1}^{-1} + 1) \frac{x_j - x}{h_j} \Big), \quad x \in I_j, \end{aligned} \tag{1.7}$$

and it follows that

$$\bar{h}(x) \leq h(x) \leq (1 + a) \bar{h}(x). \tag{1.8}$$

Similarly

$$h'(x) = r_j - r_{j-1}^{-1}, \quad x \in I_j, \tag{1.9}$$

so that

$$\|h'\|_{L_\infty} \leq a - a^{-1} \to 0 \quad \text{as } a \to 1^+. \tag{1.10}$$

1.2. **An interpolation error estimate.** We introduce the L_2-norm and the piecewise L_2-norm:

$$\begin{aligned} \|v\| &= \|v\|_{L_2(I)} = \Big(\int_0^1 v^2 \, dx \Big)^{1/2}, \\ \|v\|_{PW} &= \Big(\sum_{j=1}^n \|v\|_{L_2(I_j)}^2 \Big)^{1/2}. \end{aligned} \tag{1.11}$$

We recall the local interpolation error estimate

$$\|(v - v_I)'\|_{L_2(I_j)}^2 \leq \tfrac{1}{2} h_j^2 \|v''\|_{L_2(I_j)}^2. \tag{1.12}$$

Toghether with (1.3) this immediately implies the following two estimates in global weighted norms

$$\|(v - v_I)'\| \leq \tfrac{1}{\sqrt{2}} \|h v''\|_{PW}, \tag{1.13}$$

$$\|h^{-1}(v - v_I)'\| \leq \tfrac{1}{\sqrt{2}} \|v''\|_{PW}. \tag{1.14}$$

Of course these hold equally well with h replaced by $\bar{h}$.

1.3. **Energy norm error estimate.** Let

$$u \in V; \quad (u', v') = (f, v) \quad \forall v \in V, \tag{1.15}$$

and with $S \subset V$

$$u_S \in S; \quad (u_S', v') = (f, v) \quad \forall v \in S. \tag{1.16}$$

Then

$$(u_S' - u', v') = 0 \quad \forall v \in S. \tag{1.17}$$

and, by (1.13),

$$\|(u_S - u)'\| = \inf_{v \in S} \|(v-u)'\| \le \|(u_I - u)'\| \le \tfrac{1}{\sqrt{2}} \|h u''\|, \tag{1.18}$$

if $f \in L_2$ so that $u'' \in L_2$. This also holds with h replaced by $\bar{h}$.

1.4. **L_2-norm errror estimate.** We argue by duality. Let $e = u_S - u$. We use the dual problem

$$w \in V; \quad (v', w') = (v, e) \quad \forall v \in V. \tag{1.19}$$

Then, by taking $v = e$ and using (1.17), (1.14) and $w'' = e$,

$$\|e\|^2 = (e', w') = (e', w' - w_I') \le \|he'\| \, \|h^{-1}(w - w_I)'\| \le \|he'\| \tfrac{1}{\sqrt{2}} \|w''\| = \|he'\| \tfrac{1}{\sqrt{2}} \|e\|. \tag{1.20}$$

Hence

$$\|e\| \le \tfrac{1}{\sqrt{2}} \|he'\|. \tag{1.21}$$

If this calculation is done with h replaced by $\bar{h}$ then with (1.18) we obtain the standard (non-weighted) estimate

$$\|u_S - u\| \le \tfrac{1}{2} h_{\max}^2 \|u''\|. \tag{1.22}$$

Note: no restriction on the mesh so far.

We shall prove that

$$\|he'\| \le C \|h^2 u''\| + \tfrac{1}{2} \|e\|, \tag{1.23}$$

if $\|h'\|_{L_\infty}$ is sufficiently small (see (1.10)), and obtain from (1.21) the weighted estimate

$$\|u_S - u\| \le C \|h^2 u''\|. \tag{1.24}$$

In view of (1.8) this also holds with h replaced by $\bar{h}$ (but with a larger constant C).

To prove (1.23) we first note that

$$\|he'\|^2 = (he', he') = (e', (h^2 e)') - (2hh'e', e). \tag{1.25}$$

Note: $h \in H^1$ but $\bar{h} \notin H^1$ so we cannot use $\bar{h}$ here. Assuming

$$\|h'\|_{L_\infty} \le M, \tag{1.26}$$

we get for the last term

$$|(2hh'e', e)| \le 2M \|he'\| \, \|e\|. \tag{1.27}$$

For the first term on the right side of (1.25) we get with $v = h^2 e$, by (1.17), (1.14),

$$\begin{aligned} (e', (h^2 e)') = (e', v') = (e', v' - v_I') &\le \|he'\| \|h^{-1}(v - v_I)'\| \\ &\le \|he'\| \tfrac{1}{\sqrt{2}} \|v''\|_{PW} = \|he'\| \tfrac{1}{\sqrt{2}} \|(h^2 e)''\|_{PW}. \end{aligned} \tag{1.28}$$

On I_j we have

$$(h^2 e)'' = h^2 e'' + 4hh'e' + (2(h')^2 + 2hh'')e = h^2 u'' + 4hh'e' + 2(h')^2 e, \tag{1.29}$$

because $e'' = -u''$ and $h'' = 0$. Hence

$$\|(h^2 e)''\|_{PW} \le \|h^2 u''\| + 4M \|he'\| + 2M^2 \|e\|. \tag{1.30}$$

Inserting this and (1.27) into (1.25) and dividing by $\|he'\|$ we get

$$\|he'\| \le \tfrac{1}{\sqrt{2}} \|h^2 u''\| + \tfrac{4}{\sqrt{2}} M \|he'\| + 2(\tfrac{1}{\sqrt{2}} M^2 + M) \|e\|, \tag{1.31}$$

which implies (1.23) if M is sufficiently small. /stig