1. Weighted norm error estimates

1.1. **Notation.**

Define a mesh in the interval $I=(0,1)$:

$$
\begin{aligned}
&0=x_0<x_1<\cdots<x_{j-1}<x_j<\cdots<x_n=1,\\
&h_j=x_j-x_{j-1},\quad I_j=(x_{j-1},x_j),\quad j=1,\dots,n.
\end{aligned}
\tag{1.1}
$$

We define a piecewise linear mesh size function h by

$$
h(x_j)=h_j+h_{j+1},\ j=0,\dots,n,\quad \text{with } h_0=h_1,\ h_{n+1}=h_n. \tag{1.2}
$$

It follows that

$$
h(x)\ge h_j,\quad x\in I_j. \tag{1.3}
$$

It is important to know that h is not too large. Then we must put some restriction on the variation of the mesh. We define the piecewise constant mesh size function $\bar h$ by

$$
\bar h(x)=h_j,\quad x\in I_j, \tag{1.4}
$$

and the mesh ratios

$$
r_j=\frac{h_{j+1}}{h_j},\quad j=0,\dots,n. \tag{1.5}
$$

We assume that there is $a\ge 1$ such that

$$
r_j\in[a^{-1},a],\quad j=0,\dots,n. \tag{1.6}
$$

This is a rather weak restriction. Then

$$
\begin{aligned}
h(x)&=(h_j+h_{j+1})\frac{x-x_{j-1}}{h_j}+(h_{j-1}+h_j)\frac{x_j-x}{h_j}\\
&=h_j\Big((1+r_j)\frac{x-x_{j-1}}{h_j}+(r_{j-1}^{-1}+1)\frac{x_j-x}{h_j}\Big),\quad x\in I_j,
\end{aligned}
\tag{1.7}
$$

and it follows that

$$
\bar h(x)\le h(x)\le (1+a)\bar h(x). \tag{1.8}
$$

Similarly

$$
h'(x)=r_j-r_{j-1}^{-1},\quad x\in I_j, \tag{1.9}
$$

so that

$$
\|h'\|_{L_\infty}\le a-a^{-1}. \tag{1.10}
$$

1.2. **An interpolation error estimate.**

We introduce the L_2-norm and the piecewise L_2-norm:

$$
\begin{aligned}
\|v\|&=\|v\|_{L_2(I)}=\Big(\int_0^1 v^2\,dx\Big)^{1/2},\\
\|v\|_{PW}&=\Big(\sum_{j=1}^n\|v\|_{L_2(I_j)}^2\Big)^{1/2}.
\end{aligned}
\tag{1.11}
$$

We recall the local interpolation error estimate

$$
\|(v-v_I)'\|_{L_2(I_j)}^2\le \tfrac12 h_j^2\|v''\|_{L_2(I_j)}^2. \tag{1.12}
$$

Toghether with (1.3) this immediately implies the following two estimates in global weighted norms

$$
\|(v-v_I)'\|\le \tfrac{1}{\sqrt2}\|hv''\|_{PW}, \tag{1.13}
$$

$$
\|h^{-1}(v-v_I)'\|\le \tfrac{1}{\sqrt2}\|v''\|_{PW}. \tag{1.14}
$$

Of course these hold equally well with h replaced by $\bar h$.

1.3. **Energy norm error estimate.** Let

$$u \in V; \quad (u', v') = (f, v) \quad \forall v \in V, \tag{1.15}$$

and with $S \subset V$

$$u_S \in S; \quad (u_S', v') = (f, v) \quad \forall v \in S. \tag{1.16}$$

Then

$$(u_S' - u', v') = 0 \quad \forall v \in S. \tag{1.17}$$

and, by (1.13),

$$\|(u_S - u)'\| = \inf_{v \in S} \|(v - u)'\| \le \|(u_I - u)'\| \le \tfrac{1}{\sqrt{2}} \|hu''\|. \tag{1.18}$$

This also holds with h replaced by $\bar{h}$.

1.4. L_2**-norm errror estimate.** We argue by duality. Let $e = u_S - u$. We use the dual problem

$$w \in V; \quad (v', w') = (v, e) \quad \forall v \in V. \tag{1.19}$$

Then, by taking $v = e$ and using (1.17), (1.14) and $w'' = e$,

$$\|e\|^2 = (e', w') = (e', w' - w_I') \le \|he'\| \, \|h^{-1}(w - w_I)'\| \le \|he'\| \tfrac{1}{\sqrt{2}} \|w''\| = \|he'\| \tfrac{1}{\sqrt{2}} \|e\|. \tag{1.20}$$

Hence

$$\|e\| \le \tfrac{1}{\sqrt{2}} \|he'\|. \tag{1.21}$$

If this calculation is done with h replaced by $\bar{h}$ then with (1.18) we obtain the standard (non-weighted) estimate

$$\|u_S - u\| \le \tfrac{1}{2} h_{\max}^2 \|u''\|. \tag{1.22}$$

Note: no restriction on the mesh so far.

We shall prove that

$$\|he'\| \le C \|h^2 u''\| + \tfrac{1}{2} \|e\|, \tag{1.23}$$

if $\|h'\|_{L_\infty}$ is sufficiently small (see (1.10)), and obtain from (1.21) the weighted estimate

$$\|u_S - u\| \le C \|h^2 u''\|. \tag{1.24}$$

In view of (1.8) this also holds with h replaced by $\bar{h}$.

To prove (1.23) we first note that

$$\|he'\|^2 = (he', he') = (e', (h^2 e)') - (2hh'e', e). \tag{1.25}$$

Note: $h \in H^1$ but $\bar{h} \notin H^1$ so we cannot use $\bar{h}$ here. Assuming

$$\|h'\|_{L_\infty} \le M, \tag{1.26}$$

we get for the last term

$$|(2hh'e', e)| \le 2M \|he'\| \, \|e\|. \tag{1.27}$$

For the first term on the right side of (1.25) we get with $v = h^2 e$, by (1.17), (1.14),

$$\begin{aligned} (e', (h^2 e)') = (e', v') = (e', v' - v_I') &\le \|he'\| \|h^{-1}(v - v_I)'\| \\ &\le \|he'\| \tfrac{1}{\sqrt{2}} \|v''\|_{PW} = \|he'\| \tfrac{1}{\sqrt{2}} \|(h^2 e)''\|_{PW}. \end{aligned} \tag{1.28}$$

On I_j we have

$$(h^2 e)'' = h^2 e'' + 4hh'e' + (2(h')^2 + 2hh'')e = h^2 u'' + 4hh'e' + 2(h')^2 e, \tag{1.29}$$

because $e'' = u''$ and $h'' = 0$. Hence

$$\|(h^2 e)''\|_{PW} \le \|h^2 u''\| + 4M \|he'\| + 2M^2 \|e\|. \tag{1.30}$$

Inserting this and (1.27) into (1.25) and dividing by $\|he'\|$ we get

$$\|he'\| \le \tfrac{1}{\sqrt{2}} \|h^2 u''\| + \tfrac{4}{\sqrt{2}} M \|he'\| + 2(\tfrac{1}{\sqrt{2}} M^2 + M) \|e\|, \tag{1.31}$$

which implies (1.23) if M is sufficiently small. `/stig`