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Numerical Methods for Stochastic ODEs

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1. Lecture 1

We refer to the textbook Klebaner [3]. For the computer exercises we refer to Higham [2]. It is very important to be able to do computer simulations with stochastic ODEs, so I do recommend that you do some of these computer exercises. In these lectures I will present the theoretical background.

1.1. **Preparations.** Let X be a strong solution of the scalar Itô SDE:

$$\begin{aligned} \mathrm{d}X(t) &= \mu(X(t),t)\,\mathrm{d}t + \sigma(X(t),t)\,\mathrm{d}B(t), \quad 0 \le t \le T, \\ X(0) &= X_0. \end{aligned} \tag{1}$$

This means, see Klebaner Definition 5.1, that X solves the integral equation

$$X(t) = X_0 + \int_0^t \mu(X(s),s)\,\mathrm{d}s + \int_0^t \sigma(X(s),s)\,\mathrm{d}B(s), \quad t \in [0,T]. \tag{2}$$

Here B is Brownian motion and the last term is the Itô integral.

We assume that X_0 is a random variable and the coefficients are deterministic functions,

$$\begin{aligned} \mu &: \mathbf{R} \times [0,T] \to \mathbf{R}, \\ \sigma &: \mathbf{R} \times [0,T] \to \mathbf{R}, \end{aligned} \tag{3}$$

that are continuous and satisfy a global Lipschitz condition with respect to x,

$$\begin{aligned} |\mu(x,t) - \mu(y,t)| &\le L|x-y|, \\ |\sigma(x,t) - \sigma(y,t)| &\le L|x-y|, \end{aligned} \quad \forall x,y \in \mathbf{R},\ t \in [0,T], \tag{4}$$

and a linear growth bound,

$$\begin{aligned} |\mu(x,t)| &\le L\big(1+|x|\big), \\ |\sigma(x,t)| &\le L\big(1+|x|\big), \end{aligned} \quad \forall x \in \mathbf{R},\ t \in [0,T]. \tag{5}$$

Remark 1. The Lipschitz condition (4) is called *global* because it holds for all $x, y \in \mathbf{R}$ with the same constant L. Klebaner Theorem 5.7 assumes only a *local* Lipschitz condition, where the Lipschitz constant may depend on the size of x, y. We use a global condition in order to make the presentation simpler. Later on we will also assume a Lipschitz condition with respect to t, see (22). (There is a mistake in Theorem 5.7: the constants in (5.36) and (5.37) cannot be the same because the first one depends on N, $K = K_N$, while the second one is a global constant, independent of N.)

Remark 2. Note that (5) follows from (4) because

$$|\mu(x,t)| \le |\mu(0,t)| + |\mu(x,t) - \mu(0,t)| \le \max_{t\in[0,T]} |\mu(0,t)| + L|x-0| \le \tilde{L}\big(1+|x|\big) \tag{6}$$

with a new constant $\tilde{L}$. This is because (4) is global.

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Remark 3. Systems of SDE of the form

$$\mathrm{d}X_i = \mu_i(X,t)\,\mathrm{d}t + \sum_{j=1}^{m} \sigma_{ij}(X,t)\,\mathrm{d}B_j(t), \quad i = 1,\dots,n, \tag{7}$$

can be written in the form (1), and analyzed in the same way, with

$$\mu : \mathbf{R}^n \times [0,T] \to \mathbf{R}^n, \qquad \sigma : \mathbf{R}^n \times [0,T] \to \mathbf{R}^{n\times m},$$

and B an m-dimensional Brownian motion.

We quote Theorem 5.7 from Klebaner.

Theorem 1 (Existence and uniqueness)**.** *If X_0 is independent of $(B(t),\, 0 \le t \le T)$ and $\mathbf{E}\big(|X_0|^2\big) < \infty$, then* (1) *has a unique strong solution X and*

$$\mathbf{E}\Big(\sup_{0\le t\le T} |X(t)|^2\Big) \le C\big(1 + \mathbf{E}\big(|X_0|^2\big)\big), \tag{8}$$

where $C = C(L,T)$.

As a warm-up for the error analysis of numerical metods, we begin by proving a stability result. We will use this proof technique several times later. The bound (8) is proved in a similar way.

We need Gronwall's lemma.

Lemma 1 (Gronwall)**.** *Let A, B be constants with $B \ge 0$. If*

$$\phi(t) \le A + B\int_0^t \phi(s)\,\mathrm{d}s, \quad t \in [0,T],$$

then

$$\phi(t) \le A\mathrm{e}^{Bt}, \quad t \in [0,T].$$

Proof. $u(t) := A + \int_0^t \phi(s)\,\mathrm{d}s,\ \phi(t) \le u(t),\ u' = B\phi \le Bu,\ u(t) \le u(0)\mathrm{e}^{Bt} = A\mathrm{e}^{Bt}.$ □

We also need Doob's L_p martingale inequality, see Klebaner (7.37) p. 201.

Theorem 2 (Doob's inequality)**.** *If Y is a martingale, then*

$$\mathbf{E}\Big(\max_{0\le t\le T} |Y(t)|^p\Big) \le \Big(\frac{p}{p-1}\Big)^p \mathbf{E}\big(|Y(T)|^p\big), \quad 1 < p < \infty. \tag{9}$$

With the L_p-norm $\|X\|_{L_p} = \big(\mathbf{E}\big(|X|^p\big)\big)^{1/p} = \big(\int_\Omega |X(\omega)|^p\,\mathrm{d}\mathbf{P}(\omega)\big)^{1/p}$ this can also be written

$$\Big\| \max_{0\le t\le T} |Y(t)| \Big\|_{L_p} \le \frac{p}{p-1}\|Y(T)\|_{L_p}.$$

This gives a bound for the norm of the whole history in terms of the norm of the final value.

Finally we recall the isometry of the Itô integral, Klebaner (4.12),

$$\mathbf{E}\Big(\Big|\int_0^t f(s)\,\mathrm{d}B(s)\Big|^2\Big) = \int_0^t \mathbf{E}\big(|f(s)|^2\big)\,\mathrm{d}s. \tag{10}$$

We are now ready to prove stability with respect to perturbation of the initial value.

Theorem 3 (Stability)**.** *Let $\hat{X}$ be another solution of* (2) *with the same μ, σ, B but a different initial value $\hat{X}_0$. Then*

$$\mathbf{E}\Big(\max_{0\le t\le T} |\hat{X}(t) - X(t)|^2\Big) \le C\mathbf{E}\big(|\hat{X}_0 - X_0|^2\big), \tag{11}$$

where $C = C(L,T)$.

Note that this implies uniqueness: two solutions with the same data μ, σ, B, X_0 must be equal.

Proof. We will use Gronwall's lemma with $\phi(t) = \mathbf{E}\big(\max_{0\le s\le t} |\hat{X}(s) - X(s)|^2\big)$. By (2), writing $\hat{\mu} = \mu(\hat{X}(z), z)$, $\mu = \mu(X(z), z)$, etc, and using the inequality $(a+b+c)^2 \le 3(a^2+b^2+c^2)$, we get

$$\begin{aligned}
\phi(t) &= \mathbf{E}\Big(\max_{0\le s\le t} |\hat{X}(s) - X(s)|^2\Big) \\
&= \mathbf{E}\Big(\max_{0\le s\le t} \Big|\hat{X}_0 - X_0 + \int_0^s \big(\mu(\hat{X}(z), z) - \mu(X(z), z)\big)\,\mathrm{d}z \\
&\quad + \int_0^s \big(\sigma(\hat{X}(z), z) - \sigma(X(z), z)\big)\,\mathrm{d}B(z)\Big|^2\Big) \\
&\le 3\Big\{\mathbf{E}\Big(|\hat{X}_0 - X_0|^2\Big) + \mathbf{E}\Big(\max_{0\le s\le t}\Big|\int_0^s (\hat{\mu} - \mu)\,\mathrm{d}z\Big|^2\Big) \\
&\quad + \mathbf{E}\Big(\max_{0\le s\le t}\Big|\int_0^s (\hat{\sigma} - \sigma)\,\mathrm{d}B\Big|^2\Big)\Big\}.
\end{aligned}$$

Here, by the Cauchy-Schwartz inequality and (4),

$$\begin{aligned}
\mathbf{E}\Big(\max_{0\le s\le t}\Big|\int_0^s (\hat{\mu} - \mu)\,\mathrm{d}z\Big|^2\Big) &\le \mathbf{E}\Big(\max_{0\le s\le t}\Big\{\int_0^s 1^2\,\mathrm{d}z \int_0^s |\hat{\mu} - \mu|^2\,\mathrm{d}z\Big\}\Big) \\
&= \mathbf{E}\Big(t\int_0^t |\hat{\mu} - \mu|^2\,\mathrm{d}z\Big) = t\int_0^t \mathbf{E}\big(|\hat{\mu} - \mu|^2\big)\,\mathrm{d}s \\
&\le L^2 T \int_0^t \mathbf{E}\big(|\hat{X}(s) - X(s)|^2\big)\,\mathrm{d}s \\
&\le L^2 T \int_0^t \mathbf{E}\big(\max_{0\le z\le s} |\hat{X}(z) - X(z)|^2\big)\,\mathrm{d}s.
\end{aligned}$$

For the other term we use Doob's inequality (9) with $p = 2$ and

$$Y(t) = \int_0^t \big(\sigma(\hat{X}(z), z) - \sigma(X(z), z)\big)\,\mathrm{d}B(z),$$

which is a martingale by Theorem 4.7 in Klebaner. Using also (10) and (4) we get

$$\begin{aligned}
\mathbf{E}\Big(\max_{0\le s\le t}\Big|\int_0^s (\hat{\sigma} - \sigma)\,\mathrm{d}B\Big|^2\Big) &\le 4\mathbf{E}\Big(\Big|\int_0^t (\hat{\sigma} - \sigma)\,\mathrm{d}B\Big|^2\Big) \\
&= 4\int_0^t \mathbf{E}\big(|\hat{\sigma} - \sigma|^2\big)\,\mathrm{d}s \\
&\le 4L^2 \int_0^t \mathbf{E}\big(|\hat{X}(s) - X(s)|^2\big)\,\mathrm{d}s \\
&\le 4L^2 \int_0^t \mathbf{E}\big(\max_{0\le z\le s} |\hat{X}(z) - X(z)|^2\big)\,\mathrm{d}s.
\end{aligned}$$

We now have

$$\mathbf{E}\big(\max_{0\le s\le t} |\hat{X}(s) - X(s)|^2\big) \le 3\mathbf{E}\big(|\hat{X}_0 - X_0|^2\big) + C\int_0^t \mathbf{E}\big(\max_{0\le z\le s} |\hat{X}(z) - X(z)|^2\big)\,\mathrm{d}s$$

with $C = 3L^2(T+4)$, or

$$\phi(t) \le 3\mathbf{E}\big(|\hat{X}_0 - X_0|^2\big) + C\int_0^t \phi(s)\,\mathrm{d}s, \quad t \in [0, T],$$

where $\phi(t) = \mathbf{E}\big(\max_{0\le s\le t}|\hat{X}(s)-X(s)|^2\big)$ and Gronwall's lemma completes the proof. $\square$

Exercise 1. Without using Doob's inequality and with $\phi(t) = \mathbf{E}\big(|\hat{X}(t) - X(t)|^2\big)$ we get the weaker result

$$\mathbf{E}\big(|\hat{X}(t) - X(t)|^2\big) \le C\mathbf{E}\big(|\hat{X}_0 - X_0|^2\big), \quad t \in [0,T],$$

and hence (Prove this!)

$$\max_{0\le t\le T} \mathbf{E}\big(|\hat{X}(t) - X(t)|^2\big) \le C\mathbf{E}\big(|\hat{X}_0 - X_0|^2\big). \tag{12}$$

Note that the maximum is now outside the expected value and that (11) implies (12).

Exercise 2. Prove the estimate (8).

We also need a kind of Hölder condition.

Theorem 4. *Under the assumptions of Theorem 1 we have*

$$\mathbf{E}\big(|X(t) - X(s)|^2\big) \le C\big(1 + \mathbf{E}\big(|X_0|^2\big)\big)|t-s|, \quad \forall t,s \in [0,T], \tag{13}$$

where $C = C(L,T)$.

Proof. We may assume that $s \le t$. Working as in the previous proof, but without using Doob's inequality, we then get

$$\begin{aligned}
\mathbf{E}\big(|X(t) - X(s)|^2\big) &= \mathbf{E}\Big(\Big|\int_s^t \mu(X(z),z)\,\mathrm{d}z + \int_s^t \sigma(X(z),z)\,\mathrm{d}B(z)\Big|^2\Big) \\
&\le 2\underbrace{(t-s)}_{\le T}\int_s^t \mathbf{E}\big(|\mu|^2\big)\,\mathrm{d}z + 2\int_s^t \mathbf{E}\big(|\sigma|^2\big)\,\mathrm{d}z \quad \Big\{\text{by (5)}\Big\} \\
&\le 2L^2(T+1)\int_s^t \mathbf{E}\big((1+|X(z)|)^2\big)\,\mathrm{d}z \\
&\le 4L^2(T+1)\int_s^t \big\{1 + \mathbf{E}\big(|X(z)|^2\big)\big\}\,\mathrm{d}z \quad \Big\{\text{by (8)}\Big\} \\
&\le C(L,T)\int_s^t \big\{1 + C(L,T)\big(1+\mathbf{E}\big(|X_0|^2\big)\big)\big\}\,\mathrm{d}z \\
&\le C(L,T)(t-s)\big(1+\mathbf{E}\big(|X_0|^2\big)\big).
\end{aligned}$$

$\square$

Remark 4. Note that this means, with $\|X_0\|_{L_2} = \mathbf{E}\big(|X_0|^2\big)^{1/2} \le M$ and $C = C(M,L,T)$,

$$\|X(t) - X(s)\|_{L_2} = \Big(\mathbf{E}\big(|X(t) - X(s)|^2\big)\Big)^{1/2} \le C|t-s|^{1/2}. \tag{14}$$

This is a Hölder condition with exponent $\frac{1}{2}$. By inspection of the previous proof we see that in the deterministic case, $\sigma = 0$, we have a stronger result, a Lipschitz condition,

$$|X(t) - X(s)| \le C|t-s|. \tag{15}$$

1.2. **Strong convergence of Euler's method.** We introduce a mesh

$$0 = t_0 < t_1 < \cdots < t_n < t_{n+1} < \cdots < t_N = T, \quad h_n = t_{n+1} - t_n, \quad h = \max h_n \le 1, \tag{16}$$

and note that X satisfies

$$X(t_{n+1}) = X(t_n) + \int_{t_n}^{t_{n+1}} \mu(X(s),s)\,\mathrm{d}s + \int_{t_n}^{t_{n+1}} \sigma(X(s),s)\,\mathrm{d}B(s). \tag{17}$$

This motivates the Euler (or Euler-Maruyama) method, which defines $\{Y_n\}_{n=0}^N$ by

$$Y_0 \approx X_0, \qquad (18)$$
$$Y_{n+1} = Y_n + \mu(Y_n, t_n) \int_{t_n}^{t_{n+1}} \mathrm{d}s + \sigma(Y_n, t_n) \int_{t_n}^{t_{n+1}} \mathrm{d}B(s).$$

Since $\int_{t_n}^{t_{n+1}} \mathrm{d}s = h_n$, $\int_{t_n}^{t_{n+1}} \mathrm{d}B(s) = B(t_{n+1}) - B(t_n) = \Delta B_n$, this can also be written

$$Y_0 \approx X_0, \qquad (19)$$
$$Y_{n+1} = Y_n + \mu(Y_n, t_n)h_n + \sigma(Y_n, t_n)\Delta B_n.$$

For the practical aspects concerning the computation in a Matlab environment we refer to [2]. Here we only note that $\Delta B_n = \sqrt{h_n}\xi_n$, where the $\xi_n \in N(0,1)$ are independent Gaussian random variables with mean zero and variance 1 which can be simulated on the computer by a random number generator.

Although we only compute the node values Y_n it is convenient for our proofs to define $Y(t)$ for $t \in [t_n, t_{n+1})$ by

$$Y(t) = Y_n + \mu(Y_n, t_n) \int_{t_n}^{t} \mathrm{d}s + \sigma(Y_n, t_n) \int_{t_n}^{t} \mathrm{d}B(s), \quad t \in [t_n, t_{n+1}). \qquad (20)$$

Then Y is continuous on $[0,T], Y(t_n) = Y_n$ for all n, and

$$Y(t) = Y_0 + \int_0^t \bar{\mu}(s)\,\mathrm{d}s + \int_0^t \bar{\sigma}(s)\,\mathrm{d}B(s), \quad 0 \le t \le T, \qquad (21)$$
$$\bar{\mu}(s) = \mu(Y_n, t_n),\ \bar{\sigma}(s) = \sigma(Y_n, t_n), \quad s \in (t_n, t_{n+1}).$$

We need a Lipschitz condition with respect to t:

$$\begin{aligned} |\mu(x,t) - \mu(x,s)| &\le L|t-s|, \\ |\sigma(x,t) - \sigma(x,s)| &\le L|t-s|, \end{aligned} \quad \forall x \in \mathbf{R},\ t,s \in [0,T]. \qquad (22)$$

Theorem 5 (Strong convergence)**.** *If*

$$\mathbf{E}\big(|Y_0 - X_0|^2\big) \le Kh, \qquad (23)$$

and

$$\mathbf{E}\big(|X_0|^2\big) \le M, \qquad (24)$$

then

$$\mathbf{E}\Big(\max_{0 \le t \le T} |Y(t) - X(t)|^2\Big) \le Ch \qquad (25)$$

with $C = C(L, K, M, T)$. In particular,

$$\mathbf{E}\big(|Y_n - X(t_n)|\big) \le Ch^{1/2}, \quad n = 0, \ldots, N. \qquad (26)$$

Proof. We first prove that (25) implies (26). We use the Cauchy-Schwartz inequality,

$$|\mathbf{E}(fg)| \le \sqrt{\mathbf{E}(f^2)}\sqrt{\mathbf{E}(g^2)},$$

with $f = 1$ and $g = |Y(t_n) - X(t_n)|$:

$$\mathbf{E}\big(|Y_n - X(t_n)|\big) \le \sqrt{\mathbf{E}\big(|Y(t_n) - X(t_n)|^2\big)} \le \sqrt{\mathbf{E}\Big(\max_{0 \le t \le T} |Y(t) - X(t)|^2\Big)} \le Ch^{1/2}.$$

We now prove (25). Let $0 \le s \le t \le T$. As in the proof of Theorem 3, using (2), (21):

$$\begin{aligned}\mathbf{E}\Big(\max_{0\le s\le t}|Y(s)-X(s)|^2\Big) &= \Big\{\text{with } \mu(z)=\mu(X(z),z),\ \sigma(z)=\sigma(X(z),z)\Big\}\\ &= \mathbf{E}\Big(\max_{0\le s\le t}\Big\{\Big|Y_0-X_0+\int_0^s(\bar\mu(z)-\mu(z))\,\mathrm{d}z+\int_0^s(\bar\sigma(z)-\sigma(z))\,\mathrm{d}B(z)\Big|^2\Big\}\Big)\\ &\le 3\Big\{\mathbf{E}\big(|Y_0-X_0|^2\big)+T\int_0^t\mathbf{E}\big(|\bar\mu-\mu|^2\big)\,\mathrm{d}s+4\int_0^t\mathbf{E}\big(|\bar\sigma-\sigma|^2\big)\,\mathrm{d}s\Big\}.\end{aligned}$$

We split $\bar\mu-\mu$ into two parts and use the Lipschitz conditions (4) and (22),

$$\begin{aligned}|\bar\mu(s)-\mu(s)| &= |\mu(Y(t_n),t_n)-\mu(X(s),s)|\\ &\le |\mu(Y(t_n),t_n)-\mu(X(s),t_n)|+|\mu(X(s),t_n)-\mu(X(s),s)|\\ &\le L\big(|Y(t_n)-X(s)|+s-t_n\big)\\ &\le L\big(|Y(t_n)-X(t_n)|+|X(t_n)-X(s)|+h_n\big)\\ &\le L\big(\max_{0\le z\le s}|Y(z)-X(z)|+|X(t_n)-X(s)|+h_n\big),\quad \text{for } s\in(t_n,t_{n+1}).\end{aligned}$$

Using also (14) and $h_n \le 1$ we get

$$\begin{aligned}\mathbf{E}\big(|\bar\mu(s)-\mu(s)|^2\big) &\le 3L^2\big(\mathbf{E}\big(\max_{0\le z\le s}|Y(z)-X(z)|^2\big)+\mathbf{E}\big(|X(t_n)-X(s)|^2\big)+h_n^2\big)\\ &\le 3L^2\big(\mathbf{E}\big(\max_{0\le z\le s}|Y(z)-X(z)|^2\big)+C(s-t_n)+h_n^2\big)\\ &\le C\big(\mathbf{E}\big(\max_{0\le z\le s}|Y(z)-X(z)|^2\big)+h_n\big),\quad \text{for } s\in(t_n,t_{n+1}),\end{aligned}$$

with $C=C(M,L,T)$. The difference $\bar\sigma-\sigma$ is estimated in the same way.

Suppose that t belongs to the m-th mesh interval, $t\in(t_m,t_{m+1}]$, and denote $\tilde t_{n+1} = t_{n+1}\wedge t$ (the minimum of t and t_{n+1}). Then, since $h_n \le h$,

$$\begin{aligned}\int_0^t\mathbf{E}\big(|\bar\mu(s)-\mu(s)|^2\big)\,\mathrm{d}s &= \sum_{n=0}^m\int_{t_n}^{\tilde t_{n+1}}\mathbf{E}\big(|\bar\mu(s)-\mu(s)|^2\big)\,\mathrm{d}s\\ &\le C\sum_{n=0}^m\int_{t_n}^{\tilde t_{n+1}}\Big(\mathbf{E}\big(\max_{0\le z\le s}|Y(z)-X(z)|^2\big)+h_n\Big)\,\mathrm{d}s\\ &\le C\int_0^t\mathbf{E}\big(\max_{0\le z\le s}|Y(z)-X(z)|^2\big)\,\mathrm{d}s+Ch.\end{aligned}$$

We get the same bound for the σ-term. Therefore

$$\phi(t)\le Ch+C\int_0^t\phi(s)\,\mathrm{d}s,\quad t\in[0,T],$$

with $C=C(L,K,M,T)$, $\phi(t)=\mathbf{E}\big(\max_{0\le s\le t}|Y(s)-X(s)|^2\big)$. Gronwall's lemma completes the proof. □

We have now proved *strong convergence*, or *pathwise convergence*, of order $h^{1/2}$:

$$\Big\|\max_{0\le t\le T}|Y(t)-X(t)|\Big\|_{L_2}=\sqrt{\mathbf{E}\Big(\max_{0\le t\le T}|Y(t)-X(t)|^2\Big)}\le Ch^{1/2}. \tag{27}$$

Note that, due to the use of Gronwall's inequality, the constant in (27) grows exponentially with T, $C=\exp(C(L,K,M,T)T)$. By inspection of the previous proof we see that in the deterministic case, $\sigma=0$, we have convergence of order h:

$$\max_{0\le t\le T}|Y(t)-X(t)|\le Ch.$$

This is the classical result for Euler's method for deterministic ODEs.

In the next lecture we shall study *weak convergence* and prove that

$$\left|\mathbf{E}\big(g(Y_n) - g(X(t_n))\big)\right| \le Ch$$

for all smooth functions g. Thus the weak convergence order is h.

We shall also derive a higher order method: Milstein's method.

The presentation of some of the material in Lecture 1 was inspired by [1]. For more details see [4].

2. Lecture 2

Recall the strong solution

$$X(t) = X_0 + \int_0^t \mu(X(s),s)\,\mathrm{d}s + \int_0^t \sigma(X(s),s)\,\mathrm{d}B(s), \quad t \in [0,T], \tag{28}$$

and Euler's method

$$Y(t) = Y_0 + \int_0^t \bar{\mu}(s)\,\mathrm{d}s + \int_0^t \bar{\sigma}(s)\,\mathrm{d}B(s), \quad t \in [0,T], \tag{29}$$

where

$$\bar{\mu}(s) = \mu(Y(t_n),t_n), \quad \bar{\sigma}(s) = \sigma(Y(t_n),t_n), \quad s \in (t_n,t_{n+1}), \tag{30}$$

are piecewise constant, "frozen", functions.

We have proved *strong convergence*, or *pathwise convergence*:

$$\Big\| \max_{0\le t\le T} |Y(t)-X(t)| \Big\|_{L_2} = \sqrt{\mathbf{E}\Big(\max_{0\le t\le T} |Y(t)-X(t)|^2 \Big)} \le Ch^{1/2}. \tag{31}$$

We are often not interested in individual paths but we would like to compute the expected value of some quantity that the depends on $X(t)$, i.e., we want to compute $\mathbf{E}\big(g(X(t))\big)$ for some function g.

We shall prove that

$$\Big| \mathbf{E}\big(g(Y(t)) - g(X(t))\big) \Big| \le Ch$$

for all smooth functions g. This is called *weak convergence*. Thus the weak convergence order is h. We will use Kolmogorov's backward equation in the proof.

We shall also derive the Itô-Taylor expansion and use it to obtain a numerical method of higher order: Milstein's method.

All of this rests on Itô's formula.

2.1. Kolmogorov's backward equation.

We recall Itô's formula for functions of x,t (see Klebaner, Theorem 4.8 or Theorem 6.1). If X satisfies

$$\mathrm{d}X(t) = \mu(X(t),t)\,\mathrm{d}t + \sigma(X(t),t)\,\mathrm{d}B(t)$$

and $u = u(x,t)$ is twice differentiable in x and once in t, then (denoting the partial derivatives $u_x = \frac{\partial u}{\partial x},\ u_{xx} = \frac{\partial^2 u}{\partial x^2},\ u_t = \frac{\partial u}{\partial t}$)

$$\begin{aligned} \mathrm{d}u(X(t),t) = \Big(&u_t(X(t),t) + \mu(X(t),t)\,u_x(X(t),t) + \tfrac{1}{2}\sigma^2(X(t),t)\,u_{xx}(X(t),t) \Big)\,\mathrm{d}t \\ &+ \sigma(X(t),t)\,u_x(X(t),t)\,\mathrm{d}B(t). \end{aligned} \tag{32}$$

This really means

$$u(X(t),t) - u(X(s),s) = \int_s^t \big(u_t + \mu u_x + \tfrac{1}{2}\sigma^2 u_{xx}\big)\,\mathrm{d}z + \int_s^t \sigma u_x\,\mathrm{d}B(z), \quad 0 \le s \le t, \tag{33}$$

where the integrands are evaluated at $(X(z),z)$.

Theorem 6 (Kolmogorov's backward equation, Klebaner Theorem 6.6)**.** *Let*

$$\mathrm{d}X(t) = \mu(X(t),t)\,\mathrm{d}t + \sigma(X(t),t)\,\mathrm{d}B(t), \quad t \in [0,T], \tag{34}$$

and let $u(x,t)$ be a solution of

$$\begin{aligned} &u_t(x,t) + \mu(x,t)u_x(x,t) + \tfrac{1}{2}\sigma^2(x,t)u_{xx}(x,t) = 0, \quad x \in \mathbf{R},\ t < T, \\ &u(x,T) = g(x). \end{aligned} \tag{35}$$

Then

$$u(x,t) = \mathbf{E}\big(g(X(T)) \mid X(t) = x\big). \tag{36}$$

Note: Under appropriate assumptions we can prove that (35) has a unique solution. Then (36) is the only solution.

The notation in (36) is somewhat confusing but it is the traditional one. To explain what it means, we let $Z(s) = Z(s;x,t)$ denote the unique strong solution of

$$\begin{aligned} \mathrm{d}Z(s) &= \mu(Z(s),s)\,\mathrm{d}s + \sigma(Z(s),s)\,\mathrm{d}B(s), \quad t \le s \le T, \\ Z(t) &= x. \end{aligned}$$

In particular, the solution of (1) is $X(t) = Z(t; X_0, 0)$. Now (36) can be written

$$u(x,t) = \mathbf{E}\big(g(Z(T;x,t))\big). \tag{37}$$

Note that the variable x in (36), (37) is deterministic, while X_0 in (1) may be a random variable.

Proof. Itô's formula applied to $u(Z(s;x,t),s)$ and (35) give

$$\begin{aligned} u(Z(T;x,t),T) - u(Z(t;x,t),t) &= \int_t^T \big(u_t + \mu u_x + \tfrac{1}{2}\sigma^2 u_{xx}\big)\,\mathrm{d}s + \int_t^T \sigma u_x \,\mathrm{d}B \\ &= \int_t^T \sigma u_x \,\mathrm{d}B. \end{aligned}$$

The expected value of the Itô integral is zero, so that

$$\begin{aligned} &\mathbf{E}\big(u(Z(T;x,t),T)\big) - \mathbf{E}\big(u(Z(t;x,t),t)\big) \\ &\quad = \mathbf{E}\Big(\int_t^T \sigma(Z(s;x,t),s) u_x(Z(s;x,t),s)\,\mathrm{d}B(s)\Big) = 0. \end{aligned}$$

Since $u(Z(T;x,t),T) = g(Z(T;x,t))$ by (35) and $\mathbf{E}\big(u(Z(t;x,t),t)\big) = \mathbf{E}\big(u(x,t)\big) = u(x,t)$, we get

$$\mathbf{E}\big(g(Z(T;x,t),T)\big) = u(x,t),$$

which is (37). □

According to Definition 5.14 and Theorem 5.15 in Klebaner, there is a unique function $p(y,s,x,t),\ t < s \le T,\ x, y \in \mathbf{R}$, (a fundamental solution) such that the unique solution of (35) is given by

$$u(x,t) = \int_{-\infty}^{\infty} p(y,T,x,t) g(y)\,\mathrm{d}y, \quad x \in \mathbf{R},\ t < T. \tag{38}$$

The function $(y,s) \mapsto p(y,s,x,t)$ (with x,t fixed) satisfies the forward equation, see Klebaner (5.5a),

$$\frac{\partial p}{\partial s} + \frac{\partial}{\partial y}\Big(\mu(y,s)p\Big) - \frac{1}{2}\frac{\partial^2}{\partial y^2}\Big(\sigma^2(y,s)p\Big) = 0, \quad y \in \mathbf{R},\ t < s \le T. \tag{39}$$

The initial value at $s = t$ is the Dirac measure at x:

$$p(y,t,x,t) = \delta_x(y), \quad y \in \mathbf{R}. \tag{40}$$

In the simplified case when $\mu = 0,\ \sigma^2 = 1$, (39), (40) become

$$\begin{aligned} p_s - \tfrac{1}{2} p_{yy} &= 0, & y \in \mathbf{R},\ s > t, \\ p(y,t,x,t) &= \delta_x(y), & y \in \mathbf{R}, \end{aligned} \tag{41}$$

which is the forward heat equation with solution

$$p(y,s,x,t)=\frac{1}{\sqrt{2\pi(s-t)}}\,\mathrm{e}^{-\frac{(x-y)^2}{2(s-t)}},\quad s>t,\ x,y\in\mathbf{R},$$

the Gauss kernel. This is proved by taking the Fourier transform of (41) with respect to y. Then (38) becomes

$$u(x,t)=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi(T-t)}}\,\mathrm{e}^{-\frac{(x-y)^2}{2(T-t)}}g(y)\,\mathrm{d}y.$$

Exercise 3. Note that the partial differential equations in (39) and (35) may be written as $Lp=0$ and $L^*u=0$, respectively, where $L=\frac{\partial}{\partial t}+\frac{\partial}{\partial x}(\mu(x,t)\cdot)-\frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2(x,t)\cdot)$ and $L^*=-\frac{\partial}{\partial t}-\mu(x,t)\frac{\partial}{\partial x}-\frac{1}{2}\sigma^2(x,t)\frac{\partial^2}{\partial x^2}$. Show that the operators L and L^* are formally adjoint in the sense that

$$\int_0^T\int_{\mathbf{R}}(L\phi)\psi\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_{\mathbf{R}}\phi(L^*\psi)\,\mathrm{d}x\,\mathrm{d}t\quad\forall\phi,\psi\in C_0^\infty(\mathbf{R}\times(0,T)).\tag{42}$$

Exercise 4. Show that (38) follows from (35) and (39), (40). Hint: multiply (35) in the form $0=u_s+\mu u_y+\frac{1}{2}\sigma^2u_{yy}$ by $p(y,s,x,t)$, integrate $\int_t^T\int_{\mathbf{R}}\cdots\,\mathrm{d}y\,\mathrm{d}s$, integrate by parts and use (39), (40).

2.2. **Weak convergence of Euler's method.** In the following theorem we make rather strong assumptions on the coefficients and the initial value in order to make the proof simpler. These can be relaxed.

Theorem 7 (Weak convergence)**.** *Assume that μ,σ,g are sufficiently smooth and decay sufficiently fast as $|x|\to\infty$. Let X and Y be solutions of* (28) *and* (29) *with $Y_0=X_0$ a deterministic variable. Then*

$$\big|\mathbf{E}\big(g(Y(T))\big)-\mathbf{E}\big(g(X(T))\big)\big|\le Ch.\tag{43}$$

Proof. Let $u=u(x,t)$ be the solution of the Kolmogorov backward equation (35). Then by (36), (37) we have

$$u(x,t)=\mathbf{E}\big(g(Z(T;x,t))\big),\tag{44}$$

and, in particular, since X_0 is deterministic,

$$u(X_0,0)=\mathbf{E}\big(g(Z(T;X_0,0))\big)=\mathbf{E}\big(g(X(T))\big),\tag{45}$$

Since Y satisfies (28), Itô's formula (33) gives

$$\begin{aligned}&u(Y(T),T)-u(Y(0),0)\\&\quad=\int_0^T\Big(u_t(Y(t),t)+\bar\mu(t)u_x(Y(t),t)+\tfrac{1}{2}\bar\sigma^2(t)u_{xx}(Y(t),t)\Big)\,\mathrm{d}t\\&\qquad+\int_0^T\bar\sigma(t)u_x(Y(t),t)\,\mathrm{d}B(t).\end{aligned}$$

From (35) we get

$$u_t(Y(t),t)=-\mu(Y(t),t)u_x(Y(t),t)-\tfrac{1}{2}\sigma^2(Y(t),t)u_{xx}(Y(t),t),$$

so that

$$u(Y(T),T)-u(Y(0),0)=\int_0^T\Big((\bar\mu-\mu)u_x+\tfrac{1}{2}(\bar\sigma^2-\sigma^2)u_{xx}\Big)\,\mathrm{d}t+\int_0^T\bar\sigma u_x\,\mathrm{d}B(t),$$

where the integrands are evaluated at $(Y(t),t)$. Take expectation here, use $u(Y(T),T) = g(Y(T))$ from (35), use $Y_0 = X_0$ and (45) to get

$$\mathbf{E}\big(u(Y(0),0)\big) = \mathbf{E}\big(u(Y_0,0)\big) = \mathbf{E}\big(u(X_0,0)\big) = \mathbf{E}\big(g(X(T))\big),$$

and use $\mathbf{E}\big(\int_0^T \bar\sigma u_x \,\mathrm{d}B\big) = 0$. We get

$$\begin{aligned}\mathbf{E}\big(g(Y(T))\big) - \mathbf{E}\big(g(X(T))\big) &= \int_0^T \Big\{\mathbf{E}\big((\bar\mu-\mu)u_x\big) + \tfrac12\mathbf{E}\big((\bar\sigma^2-\sigma^2)u_{xx}\big)\Big\}\,\mathrm{d}t\\ &= \sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} \Big\{\mathbf{E}\big((\bar\mu-\mu)u_x\big) + \tfrac12\mathbf{E}\big((\bar\sigma^2-\sigma^2)u_{xx}\big)\Big\}\,\mathrm{d}t.\end{aligned}$$

We shall show

$$\sup_{t\in(t_n,t_{n+1})}\Big|\mathbf{E}\big((\bar\mu(t)-\mu(Y(t),t))u_x(Y(t),t)\big)\Big| \le Ch_n, \tag{46}$$

$$\sup_{t\in(t_n,t_{n+1})}\tfrac12\Big|\mathbf{E}\big((\bar\sigma^2(t)-\sigma^2(Y(t),t))u_{xx}(Y(t),t)\big)\Big| \le Ch_n, \tag{47}$$

which implies the desired result.

To prove (46) we define

$$v(x,t) = \big(\mu(Y_n,t_n)-\mu(x,t)\big)u_x(x,t),$$

and note that (46) means (recall that $\bar\mu(t) = \mu(Y(t_n),t_n)$ for $t\in(t_n,t_{n+1})$)

$$\Big|\mathbf{E}\big(v(Y(t),t)\big)\Big| \le Ch_n, \quad t\in(t_n,t_{n+1}).$$

Since Y satisfies (28), Itô's formula (33) gives

$$\begin{aligned}&v(Y(t),t) - v(Y(t_n),t_n)\\ &\quad= \int_{t_n}^t \Big(v_t(Y(t),t) + \bar\mu(t)v_x(Y(t),t) + \tfrac12\bar\sigma^2(t)v_{xx}(Y(t),t)\Big)\,\mathrm{d}t\\ &\qquad+ \int_{t_n}^t \bar\sigma(t)v_x(Y(t),t)\,\mathrm{d}B(t).\end{aligned}$$

Here $v(Y(t_n),t_n) = 0$ and after taking expectation we get

$$\begin{aligned}\big|\mathbf{E}\big(v(Y(t),t)\big)\big| &= \Big|\int_{t_n}^t \mathbf{E}\Big(v_t(Y(t),t) + \bar\mu(t)v_x(Y(t),t) + \tfrac12\bar\sigma^2(t)v_{xx}(Y(t),t)\Big)\,\mathrm{d}t\Big|\\ &\le h_n \sup_{t\in(t_n,t_{n+1})}\Big(|v_t(Y(t),t)| + |\bar\mu(t)|\,|v_x(Y(t),t)| + \tfrac12|\bar\sigma^2(t)|\,|v_{xx}(Y(t),t)|\Big),\end{aligned}$$

which proves (46) provided that

$$\begin{aligned}&\max_{\mathbf{R}\times[0,T]}|\mu(x,t)| \le C,\\ &\max_{\mathbf{R}\times[0,T]}|\sigma(x,t)| \le C,\\ &\max_{\mathbf{R}\times[0,T]}\big(|v_t(x,t)| + |v_x(x,t)| + |v_{xx}(x,t)|\big) \le C,\end{aligned}$$

where the last one in its turn follows from the bounds

$$\begin{aligned}&\max_{\mathbf{R}\times[0,T]}(|\mu_t(x,t)| + |\mu_x(x,t)| + |\mu_{xx}(x,t)|) \le C,\\ &\max_{\mathbf{R}\times[0,T]}(|u_{xt}(x,t)| + |u_x(x,t)| + |u_{xx}(x,t)| + |u_{xxx}(x,t)|) \le C.\end{aligned}$$

Such bounds can be proved provided that μ, σ, g are sufficiently smooth and decay sufficiently fast as $|x| \to \infty$. This proves (46), and (47) is proved in the same way. $\square$

This proof was inspired by [5].

If X_0 is a random variable, then we must first show

$$u(X_0, 0) = \mathbf{E}\big(g(X(T)) \mid X_0\big),$$

so that (by the law of double expectation)

$$\mathbf{E}\big(u(X_0, 0)\big) = \mathbf{E}\big(\mathbf{E}\big(g(X(T)) \mid X_0\big)\big) = \mathbf{E}\big(g(X(T))\big)$$

instead of (45). The proof can then be continued in the same way. We do not the present the details.

2.3. **Sampling error.** In practice we compute several sample paths $\{Y(t, \omega_j)\}_{j=1}^M$, $\omega_j \in \Omega$, and approximate the expected value $\mathbf{E}\big(g(X(T))\big)$ by the average $\frac{1}{M}\sum_{j=1}^M g(Y(T, \omega_j))$. The total error is then the sum of the *discretization error* and the *statistical error (sampling error)*:

$$\begin{aligned} &\Big|\frac{1}{M}\sum_{j=1}^M g(Y(T, \omega_j)) - \mathbf{E}\big(g(X(T))\big)\Big| \\ &\quad \le \Big|\mathbf{E}\big(g(Y(T))\big) - \mathbf{E}\big(g(X(T))\big)\Big| + \Big|\frac{1}{M}\sum_{j=1}^M \Big(g(Y(T, \omega_j)) - \mathbf{E}\big(g(Y(T))\big)\Big)\Big|. \end{aligned}$$

We have shown that the first part is $\le Ch$, and by using the *central limit theorem* it can be shown that the sampling error is $\le \frac{C}{\sqrt{M}}$. See [5].

2.4. **A numerical method of higher order: Milstein's method.** We derive the first steps of the *Itô-Taylor expansion*. We consider for simplicity the autonomous equation

$$\mathrm{d}X(t) = \mu(X(t))\,\mathrm{d}t + \sigma(X(t))\,\mathrm{d}B(t),$$

where $\mu = \mu(x), \sigma = \sigma(x)$ do not depend on t. Then

$$X(t) = X(t_0) + \int_{t_0}^t \mu(X(s))\,\mathrm{d}s + \int_{t_0}^t \sigma(X(s))\,\mathrm{d}B(s). \tag{48}$$

Itô's formula is, see Klebaner (4.53),

$$\begin{aligned} f(X(t)) &= f(X(t_0)) + \int_{t_0}^t \Big(\mu(X(s))f'(X(s)) + \frac{1}{2}\sigma^2(X(s))f''(X(s))\Big)\,\mathrm{d}s \\ &\quad + \int_{t_0}^t \sigma(X(s))f'(X(s))\,\mathrm{d}B(s) \\ &= f(X(t_0)) + \int_{t_0}^t L_0 f(X(s))\,\mathrm{d}s + \int_{t_0}^t L_1 f(X(s))\,\mathrm{d}B(s), \end{aligned} \tag{49}$$

where the differential operators L_0 and L_1 are defined by

$$\begin{aligned} L_0 f &= \mu f' + \frac{1}{2}\sigma^2 f'', \\ L_1 f &= \sigma f'. \end{aligned}$$

In particular, with $f(x) = x$ we get $L_0 f = \mu$, $L_1 f = \sigma$ and we retrieve (48). Next we take $f = \mu$ and $f = \sigma$ in (49) and insert the result into (48). We get

$$\begin{aligned} X(t) = X(t_0) &+ \int_{t_0}^{t} \Big(\mu(X(t_0)) + \int_{t_0}^{s} L_0\mu(X(z))\,\mathrm{d}z + \int_{t_0}^{s} L_1\mu(X(z))\,\mathrm{d}B(z) \Big)\,\mathrm{d}s \\ &+ \int_{t_0}^{t} \Big(\sigma(X(t_0)) + \int_{t_0}^{s} L_0\sigma(X(z))\,\mathrm{d}z + \int_{t_0}^{s} L_1\sigma(X(z))\,\mathrm{d}B(z) \Big)\,\mathrm{d}B(s) \\ &= X(t_0) + \mu(X(t_0)) \int_{t_0}^{t} \mathrm{d}s + \sigma(X(t_0)) \int_{t_0}^{t} \mathrm{d}B(s) + R_1(t, t_0), \end{aligned}$$

where the remainder is

$$\begin{aligned} R_1(t, t_0) = &\int_{t_0}^{t} \int_{t_0}^{s} L_0\mu(X(z))\,\mathrm{d}z\,\mathrm{d}s \\ &+ \int_{t_0}^{t} \int_{t_0}^{s} L_1\mu(X(z))\,\mathrm{d}B(z)\,\mathrm{d}s \\ &+ \int_{t_0}^{t} \int_{t_0}^{s} L_0\sigma(X(z))\,\mathrm{d}z\,\mathrm{d}B(s) \\ &+ \int_{t_0}^{t} \int_{t_0}^{s} L_1\sigma(X(z))\,\mathrm{d}B(z)\,\mathrm{d}B(s). \end{aligned}$$

This is the motivation for Euler's method.

Next use Itô's formula with $f = L_1\sigma = \sigma\sigma'$ and insert into the last term in the remainder to get

$$\begin{aligned} X(t) = X(t_0) &+ \mu(X(t_0)) \int_{t_0}^{t} \mathrm{d}s + \sigma(X(t_0)) \int_{t_0}^{t} \mathrm{d}B(s) \\ &+ \sigma(X(t_0))\sigma'(X(t_0)) \int_{t_0}^{t} \int_{t_0}^{s} \mathrm{d}B(z)\,\mathrm{d}B(s) + R_2(t, t_0), \end{aligned}$$

where the new remainder $R_2(t, t_0)$ consists of five complicated terms.

This is the motivation for Milstein's method. We have here

$$\int_{t_0}^{t} \mathrm{d}s = t - t_0, \quad \int_{t_0}^{t} \mathrm{d}B(s) = B(t) - B(t_0),$$

and the iterated Itô integral, see Klebaner Example 4.12 on p. 107,

$$\int_{t_0}^{t} \int_{t_0}^{s} \mathrm{d}B(z)\,\mathrm{d}B(s) = \int_{t_0}^{t} \Big(B(s) - B(t_0) \Big)\,\mathrm{d}B(s) = \frac{1}{2} \Big(\big(B(t) - B(t_0) \big)^2 - (t - t_0) \Big).$$

The Milstein method defines $\{Y_n\}_{n=0}^{N}$ by $Y_0 = X_0$ and

$$Y_{n+1} = Y_n + \mu(Y_n)h_n + \sigma(Y_n)\Delta B_n + \tfrac{1}{2}\sigma(Y_n)\sigma'(Y_n)\big((\Delta B_n)^2 - h_n \big),$$

where

$$h_n = t_{n+1} - t_n, \quad \Delta B_n = B(t_{n+1}) - B(t_n).$$

It can be shown that it converges with strong order h:

$$\mathbf{E}\big(|Y_n - X(t_n)| \big) \le Ch.$$

This idea can be extended to derive methods of higher order but these are not very useful in practice because the coefficients are many times iterated integrals, for example, $\int_{t_0}^{t} \cdots \int_{t_0}^{z_2} \mathrm{d}B(z_1) \cdots \mathrm{d}B(z_n)\,\mathrm{d}s_1 \cdots \mathrm{d}s_k$, which are difficult to evaluate.

More material on iterated Itô formulas and Itô-Taylor expansions can be found in [4].

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