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Numerical Methods for Stochastic ODEs

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Assignment

1. Existence. Prove Theorem 1. That is, show that there is a unique strong solution on the interval $[0, T]$.

Hint: note that equation (2) is a fixed point equation $X = G(X)$ and show that the operator

$$G(Y)(t) = X_0 + \int_0^t \mu(Y(s), s)\,\mathrm{d}s + \int_0^t \sigma(Y(s), s)\,\mathrm{d}B(s),$$

is a contraction on the Banach space

$$W_{[0,\tau]} = \Big\{ Y \in C([0, \tau], L_2(\Omega)) : Y \text{ is adapted to the filtration generated by } B \Big\}$$

with norm

$$\|Y\|_{W_{[0,\tau]}} = \max_{0 \le t \le \tau} \|Y(t)\|_{L_2} = \max_{0 \le t \le \tau} \sqrt{\mathbf{E}\big(|Y(t)|^2\big)}$$

provided that τ is small enough. Hence, we obtain a solution on a (short) interval $[0, \tau]$. Repeat and obtain solutions on $[\tau, 2\tau]$, $[2\tau, 3\tau]$, and so on until we cover the whole interval $[0, T]$.

2. Matlab computations. Read Higham and do (at least) the programs `bpath1.m`, `bpath2.m`, `bpath3.m`, `stint.m`, `em.m`, `emstrong.m`.

3. Black-Scholes process.

$$\begin{aligned} \mathrm{d}X(t) &= rX(t)\,\mathrm{d}t + \sigma X(t)\,\mathrm{d}B(t) \\ X(0) &= X_0 \end{aligned}$$

with constants $r > 0$, $\sigma > 0$. (a) Show that the unique strong solution is given by the formula

$$X(t) = X_0 \exp((r - \tfrac{1}{2}\sigma^2)t + \sigma B(t))$$

(b) Write a Matlab program and solve the equation by the Euler method. Plot several sample paths and compare with the solution formula. Examine strong convergence. (This is `emstrong.m`.)

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4. Ornstein-Uhlenbeck process. The Langevin equation is

$$
\begin{aligned}
\mathrm{d}X(t) &= -\alpha X(t)\,\mathrm{d}t + \sigma\,\mathrm{d}B(t)\\
X(0) &= X_0
\end{aligned}
$$

with constants $\alpha > 0$, $\sigma > 0$. (a) Show that the unique strong solution is given by the formula

$$
X(t) = \mathrm{e}^{-\alpha t} X_0 + \sigma \int_0^t \mathrm{e}^{-\alpha(t-s)}\,\mathrm{d}B(s)
$$

(b) Write a Matlab program and solve the equation by the Euler method. Plot several sample paths and compare with the solution formula. Examine strong convergence.