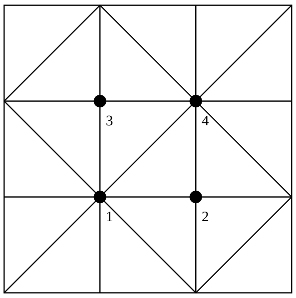
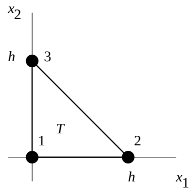
TMA371 Partial Differential Equations TM, 1997–12–16. Solutions.

1. Let V_h be the usual finite element space consisting of continuous piecewise linear functions satisfying the boundary condition $v = 0$ on $\partial\Omega$. The cG(1) method is

$$
\begin{aligned}
&U \in V_h, \\
&(\nabla U, \nabla v) + (U, v) = (f, v) \quad \forall v \in V_h.
\end{aligned}
$$





Making the "Ansatz" $U(x) = \sum_{i=1}^{4} \xi_i \varphi_i(x)$, where φ_i are the standard basis functions, we obtain the system of equations

$$\sum_{i=1}^{4} \xi_i \Big(\int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \, dx + \int_\Omega \varphi_i \varphi_j \, dx \Big) = \int_\Omega f \varphi_j \, dx, \quad j = 1, \dots 4,$$

or, in matrix form,

$$(K + M)\xi = F,$$

where $K_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$ is the stiffness matrix, $M_{ij} = (\varphi_i, \varphi_j)$ is the mass matrix, and $F_j = (f, \varphi_j)$ is the load vector.

We first compute the mass and stiffness matrices for the reference triangle T. The local basis functions are

$$
\begin{aligned}
\phi_1(x_1, x_2) &= 1 - \frac{x_1}{h} - \frac{x_2}{h}, & \nabla\phi_1(x_1, x_2) &= -\frac{1}{h}\begin{bmatrix}1\\1\end{bmatrix}, \\
\phi_2(x_1, x_2) &= \frac{x_1}{h}, & \nabla\phi_2(x_1, x_2) &= \frac{1}{h}\begin{bmatrix}1\\0\end{bmatrix}, \\
\phi_3(x_1, x_2) &= \frac{x_2}{h}, & \nabla\phi_3(x_1, x_2) &= \frac{1}{h}\begin{bmatrix}0\\1\end{bmatrix}.
\end{aligned}
$$

Hence, with $|T| = \int_T dx = h^2/2$,

$$
\begin{aligned}
m_{11} &= (\phi_1, \phi_1) = \int_T \phi_1^2 \, dx = h^2 \int_0^1 \int_0^{1-x_2} (1 - x_1 - x_2)^2 \, dx_1 \, dx_2 = \frac{h^2}{12}, \\
k_{11} &= (\nabla\phi_1, \nabla\phi_1) = \int_T |\nabla\phi_1|^2 \, dx = \frac{2}{h^2}|T| = 1.
\end{aligned}
$$

Alternatively, we can use the midpoint rule, which is exact for polynomials of degree 2 (precision 3):

$$m_{11} = (\phi_1, \phi_1) = \int_T \phi_1^2 \, dx = \frac{|T|}{3} \sum_{j=1}^{3} \phi_1(\hat{x}_j)^2 = \frac{h^2}{6}\Big(0 + \frac{1}{4} + \frac{1}{4}\Big) = \frac{h^2}{12},$$

where $\hat{x}_j$ are the midpoints of the edges. Continuing like this, we easily compute

$$m = \frac{h^2}{24}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad k = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We can now assemble the global matrices M, K from the local matrices m, k:

$$\begin{aligned}
&M_{11} = M_{44} = 8m_{22} = \frac{8}{12}h^2, && K_{11} = K_{44} = 8k_{22} = 4, \\
&M_{12} = M_{13} = M_{24} = M_{34} = 2m_{12} = \frac{1}{12}h^2, && K_{12} = K_{13} = K_{24} = K_{34} = 2k_{12} = -1, \\
&M_{14} = 2m_{23} = \frac{1}{12}h^2, && K_{14} = 2k_{23} = 0, \\
&M_{22} = M_{33} = 4m_{11} = \frac{4}{12}h^2, && K_{22} = K_{33} = 4k_{11} = 4, \\
&M_{23} = 0, && K_{23} = 0.
\end{aligned}$$

The remaining matrix elements are obtained by symmetry $M_{ij} = M_{ji}$, $K_{ij} = K_{ji}$. Hence,

$$M = \frac{h^2}{12}\begin{bmatrix} 8 & 1 & 1 & 1 \\ 1 & 4 & 0 & 1 \\ 1 & 0 & 4 & 1 \\ 1 & 1 & 1 & 8 \end{bmatrix}, \quad K = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}.$$

2. The problem can be written

$$u \in V; \quad a(u,v) = (f,v) \quad \forall v \in V,$$

and the cG(1) problem is

$$U \in V_h; \quad a(U,v) = (f,v) \quad \forall v \in V_h,$$

where $V = H_0^1(0,1)$ and $a(u,v) = (u',v') + (u,v)$. Hence, we have the orthogonality

$$a(u-U,v) = 0 \quad \forall v \in V_h.$$

We prove an a priori error estimate in the energy norm. The energy norm is defined by $\|v\|_E^2 = \|v'\|^2 + \|v\|^2 = a(v,v)$. Hence, with $e = u - U$ and the standard interpolant $\pi_h u \in V_h$,

$$\begin{aligned}
\|e\|_E^2 &= a(e,e) \\
&= a(e, u - \pi_h u) + a(e, \pi_h u - U) \\
&= a(e, u - \pi_h u) \quad \text{(orthogonality)} \\
&\le \|e\|_E \, \|u - \pi_h u\|_E, \quad \text{(Cauchy's inequality)}
\end{aligned}$$

so that

$$\|e\|_E \le \|u - \pi_h u\|_E.$$

The interpolation error estimates give

$$\begin{aligned}
\|(u - \pi_h u)'\| &\le C_1 \|h u''\| \\
\|u - \pi_h u\| &\le C_2 \|h^2 u''\| \le C_2 h_{\max} \|h u''\| \le C_3 \|h u''\|,
\end{aligned}$$

so that $\|u - \pi_h u\|_E \le \sqrt{C_1^2 + C_3^2}\|hu''\| = C\|hu''\|$ and $\|e\|_E \le C\|hu''\|$.

3. (Göteborg University). See the book.

3. (Chalmers). (a) Multiply the equation by $\dot u$:

$$
\begin{aligned}
&(\ddot u, \dot u) - (\Delta u, \dot u) + (u, \dot u) = 0,\\
&(\ddot u, \dot u) + (\nabla u, \nabla \dot u) + (u, \dot u) = 0,\\
&\frac{1}{2}\frac{d}{dt}\big(\|\dot u\|^2 + \|\nabla u\|^2 + \|u\|^2\big) = 0,\\
&\frac{1}{2}\big(\|\dot u(t)\|^2 + \|\nabla u(t)\|^2 + \|u(t)\|^2\big) = \frac{1}{2}\big(\|u_1\|^2 + \|\nabla u_0\|^2 + \|u_0\|^2\big).
\end{aligned}
$$

This means that the energy $E(t) = \frac{1}{2}\big(\|\dot u(t)\|^2 + \|\nabla u(t)\|^2 + \|u(t)\|^2\big)$ is conserved.

(b) Set $v_1 = \dot u$, $v_2 = u$. Then

$$
\begin{aligned}
&\dot v_1 - \Delta v_2 + v_2 = 0,\\
&\dot v_2 - v_1 = 0.
\end{aligned}
$$

Now we have a system $\dot v + Av = 0$ of first order in t and we can use various techniques developed for such systems, for example, we can apply standard time-discretization methods such as dG(0) or cG(1).

4. (a) $u(t) = e^{-A(t)}u_0, \quad A(t) = \int_0^t a(s)\,ds$, so that

$$
\begin{aligned}
\int_0^T |\dot u(t)|\,dt &= \int_0^T |a(t)e^{-A(t)}u_0|\,dt\\
&= \int_0^T a(t)e^{-A(t)}\,dt\,|u_0|\\
&= \Big(1 - e^{-A(T)}\Big)|u_0| \le |u_0|,
\end{aligned}
$$

where we used the assumption that $a(t) \ge 0$, and hence also $A(t) \ge 0$.

(b) See the book.

5. (a) We have $\Gamma_- = \{x \in \Gamma : \beta(x)\cdot n(x) < 0\}$ and $\Gamma_+ = \{x \in \Gamma : \beta(x)\cdot n(x) \ge 0\}$. Multiply the equation by u:

$$(\beta\cdot\nabla u, u) + (\alpha u, u) = (f, u). \tag{1}$$

Green's formula gives

$$
\begin{aligned}
(\beta\cdot\nabla u, u) &= \int_\Gamma \beta\cdot n\,u^2\,ds - \int_\Omega u\nabla\cdot(\beta u)\,dx\\
&= \int_\Gamma \beta\cdot n\,u^2\,ds - \int_\Omega u\big(\nabla\cdot\beta u + \beta\cdot\nabla u\big)\,dx\\
&= \int_{\Gamma_+} \beta\cdot n\,u^2\,ds + \int_{\Gamma_-} \beta\cdot n\,u^2\,ds - (\nabla\cdot\beta u, u) - (\beta\cdot\nabla u, u),
\end{aligned}
$$

so that, since $\beta\cdot n < 0$ and $u = g$ on Γ_-,

$$2(\beta\cdot\nabla u, u) = \int_{\Gamma_+} \beta\cdot n\,u^2\,ds - \int_{\Gamma_-} |\beta\cdot n|\,g^2\,ds - (\nabla\cdot\beta u, u).$$

We insert this into (1):

$$\int_{\Gamma_+} \beta\cdot n\,u^2\,ds + 2\big((\alpha - \tfrac{1}{2}\nabla\cdot\beta)u, u\big) = 2(f, u) + \int_{\Gamma_-} |\beta\cdot n|\,g^2\,ds. \tag{2}$$

Here we have $\alpha - \frac{1}{2}\nabla\cdot\beta \ge c > 0$ and

$$2(f, u) \le 2\|f\|\,\|u\| = 2\big(c^{-1/2}\|f\|\big)\big(c^{1/2}\|u\|\big) \le \frac{1}{c}\|f\|^2 + c\|u\|^2,$$

so that (2) becomes

$$\int_{\Gamma_+} \beta \cdot n\, u^2\, ds + 2c\|u\|^2 \leq \frac{1}{c}\|f\|^2 + c\|u\|^2 + \int_{\Gamma_-} |\beta \cdot n|\, g^2\, ds.$$

This implies

$$\int_{\Gamma_+} \beta \cdot n\, u^2\, ds + c\|u\|^2 \leq \frac{1}{c}\|f\|^2 + \int_{\Gamma_-} |\beta \cdot n|\, g^2\, ds.$$

(b) See the book.

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