TMA371 Partial Differential Equations 1997
Assignment 2b

1. Consider the heat equation:

$$\begin{aligned} u_t - \Delta u &= 0, \quad x \in \Omega,\ t > 0, \\ u &= 0 \qquad x \in \partial\Omega,\ t > 0, \\ u(x,0) &= u_0 \quad x \in \Omega. \end{aligned}$$

a) Show the following stability estimates:

$$\|u(t)\|^2 + \int_0^t \|\nabla u\|^2 \, ds \le \|u_0\|^2, \quad t \ge 0,$$

$$\|\Delta u(t)\| \le \frac{1}{t}\|u_0\|, \quad t > 0.$$

The latter estimate is the so called parabolic smoothing estimate (or strong stability), that describes the fact that the solution is smoother than the initial value.

b) Solve the problem with $\Omega = [0,1]$ using a Fourier series and study how fast the coefficients for the different Fourier modes decay. Prove the smoothing estimate by using this Fourier series representation of the solution.

2. Solve Problems 17.27 and 17.35 in the book. Hint for 17.35: observe that $w_j = k_n^{-1} \int_{I_n} U_j \, dt$ at the top of page 443.

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