*A commentary on*

**Orchestring a Mathematical theme:
Eleven-year olds discuss the problem
of Infinity**

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The context: A few groups of eleven-year olds (from now one refered to by the quaint word of 'pupils') have been given the practical task of cutting a board into two, three or four pieces of equal length. However they are not given the implements appropriate of such a task, as there are no saws, rulers, or whatever present. Thus they are given to understand that their assignment is 'theoretical'. The conclusion they seem to consider foregone, is to convert the fractions $1/2, 1/4$ and $1/3$ into decimal representations. This is a mathematical task. A natural question to ponder is why this mathematical task is appropriate, and more to the point why the pupils think so.

In the absence of documentation one is led to speculation. A natural suspicion is the concept of a ruler, which in our culture shows decimeters, centimeters etc[1] . The problem is then to convert the desired lengths of the pieces, to a point on the ruler, identified by a (simple) decimal fraction. The ruler is not present, except in the imagination of the pupils. One gets the impression that they consider a ruler, the exact length of the given board, although this assumption is probably tacit. Thus as they are familiar with the representation of a half as 50% and a quarter as 25% they tend to think of the problem as being solved in those two cases, however the problem of three equal parts, presents a challenge. Why?

At this point it would be interesting to know the exact preparation of the pupils. This is hard to access, once you go beyond the documented curriculum, as decimal representations of numbers, usually as regards to percentages, are ubiquitous in newspapers[2]. However one surmises that the pupils know the concept of dividing numbers with numbers, and that this is not always possible, and that the 'obstruction' for a successful division is refered to as the remainder. The remainder is usually thought of as a failure, something to be fudged away. The impossibility of integral division must be very familiar to the students. Here they have a new example, dividing 100 by three. Why a hundred? Obviously it must be the percentage notion. Expressing something in percentage is just a matter of writing something as a fraction with 100 in the denominator. It is not clear to me that most pupils (or adults for that matter) are really aware of this, in spite of the terminology *per cent.* Dividing by three they come up implicitly with the annoying remainder, and hence the temptation to fudge. (The example of 33, 33 and 34, as being accurate enough for the task at hand). The fact that $1/2$ is fifty percent seems to be something most pupils learn by themselves. In fact dealing with halves, seems to be something rather natural to most people, as opposed to other fractions. To divide something in two equal parts seems unproblematic, after all the two parts are equal and add up. But to divide by three seems more problematic for some reason. The almost tautological definition of a half, does not seem to carry as much conviction in the case of a third.[3] It is not clear whether this is a

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[1] There are of course geometric ways of dividing lengths in parts, but such do not seem to be part of the imagination of most pupils. Maybe they would not even recognise such as mathematical solutions, would they be presented with them. As I am unaware of the history and culture of carpentry throught he ages, I cannot dwell on this in depth, although I would like to think that pupils at the time of Euclide, would have approached this problem geometrically

[2] One is reminded of jurors in the American legal system, who are chosen with the express purpose of being unknowledgable of the crime in question, and thereafter being kept in isolation, to prevent being influenced by outside media coverage

[3] The formal mathematical definition of fractions as an extension of the integers, by solutions to equations of type $px = q$, makes the notion of a fraction indeed tautological by definition, but it enables the notion of multiplication and addition, as well as their inverses, which is far from obvious. However, we do not think of fractions as something we create, but actually as something that actually do exist. Fractions do not make sense when counting (but a half-apple seems to be better defined than a third-apple) but only in matter of continua (lengths, mass etc)

deep-seated feature of our brain (we are after all bi-symmetrical beings, and in mathematics and especially number-theory, the prime two stands out, and almost always needs special treatment) or a consequence of the decimal notion. On the other hand we are all aware of 30 minutes as being half an hour, but we seldom think of thirds of hours, in particular not that 20 minutes is a third of an hour. As halfing is natural, it should be inductively applied, but it seems to take a bit more sophistication to realise that a quarter is a half of a half, and hence that the sum of two quarters is a half. In decimal countries, the further continuation, i.e. working with eights, and sixteenths, is not part of everyday life; I wonder though about U.S.A and their antiquated system of measures, that is built upon halfing. (Gallons, quarts thereof, pints, cups..., as well as the division of the yard-stick).

At this stage it would be of paramount interest to the reader to know what the students have been formally taught. I recall that the introduction of decimal expansions was not taught at my time until the fifth grade. (Which in retrospect seems rather remarkable). By the algorithms of decimal expansion, the pupils finally are set to deal with the annoying remainder.[4] If the pupils have already been taught this algorithm, it is inevitable that they must have encountered the problem of dividing 1 by three, and seeing that this leads to an interminable sequence. In that case their re-discovery of that phenomenon, is hardly a source for amazement. It could be, however, that the students have never been told how to convert fractions into decimal fractions, but nevertheless having some familiarity with the manipulation of such, and seen decimal representations of numbers. Maybe they have even seen $\pi$ expressed with many decimals, something that may intrigue most mathematically curious students. For the documentation of the paper to be interesting, we need to accept this as an assumption. (However it is not clear to me how to do this rigorously, the naive method of asking the students whether they are aware of this and that, obviously will make them aware, and the interrogation becomes part of the problem.).

At this stage it might be appropriate to discuss, why we use decimal expansions[5], and also possibly to inquire into its history. The positional system allows an economy of notation as all strings (of digits) are allowed, and there is no duplication (with the convention of dropping initial zeroes). Furthermore it makes comparison of integers obvious, as well as allowing convenient arithmetical algorithms. It is part of our culture, and not inate, hence making instruction both imperative and necessary.

Fractions are a mess. First it is not always so clear to compare two fractions, which one is bigger, which one is smaller. There is a simple algorithm for doing so, by essentially putting on a common denominator. The multiplication and division of fractions is rather simple, and to the child it comes as a surprise that division is as easy as multiplication.[6]. The summing of fractions is however different, you need to know the trick of common denominators.[7] And even if you know it, the summing of a few innocous fractions quickly becomes unwieldy as the common denominator can grow quite quickly. (In fact exponentially in terms of the number of terms.) A handier way is required.

[4] The powerful method of working with modular aritmetic, only retaining the remainder, is never taught in elementary school. Whether this is a wise thing or not I do not know. Many pupils do not encounter it until their university education, which is rather remarkable. But I may be mistaken on this count.

[5] What really is important is the positional system, the choice of base is arbitrary, and from a mathematical point of view uninteresting. The possible exception being the base of two, whose use in computers every reader is well aware of

[6] I recall personally that it came as a surprise to me that division of fractions was the same thing as multiplying with the inverted fraction. Why this was so, was not explained, and it was a mystery to me. Why it was a mystery to me at the time, is now a mystery to me. I have a faint recollection that prior to this I instinctively 'knew' how to handle compound fractions (say $\frac{\frac{1}{3}}{\frac{1}{2}}$), at least if not the numbers involved were not too intimidatingly big, but it must not have occured to me that what I did was just multiplying with the inverted fraction. Just another possible instance of compartimentalization, not being aware in a slightly different context, that you actually know how to handle things

[7] This 'trick' was not taught in elementary school at least not in the fifties and early sixties, and I doubt that things have changed since then. In fact I would not be surprised that the adding of fractions is now altogether removed from the school curriculum, and maybe just a minority of adults know how to do it. I did not know how to do it in sixth grade, the trick was pointed out to me, and I was very embarrassed that I had not 'thought' of it myself.

The convention of percentages points to the way. The point of percentages is to denote fractions of unity[8]in terms of an (integral) number of hundreds. For many practical purposes, one hundredth is accurate enough. However there are minute fractions leading to *pro mille*[9], and in scientific work one starts using prefixes as mille, mikro, nano etc. The solution to refined accuracies is to use a dynamical stratagem. Rather than restricting the denominators to a hundred, in principle any power of ten will be allowed in the denominator. A decimal fraction is thus simply a fraction of type

$$\frac{141421356237}{100000000000}$$

Writing so many zeroes is a waste of space and effort, why not just keep track of their number, and write say

$$141421356237[11]$$

I do not know, but I very much doubt it, whether such an intermediate notation ever existed historically, although an equivalent one must be used for the coding of floating numbers in computers.[10]. The solution is, as we all know, to insert at the appropriate spot a point (or a comma, or whatever the convention).

$$1.41421356237$$

In this way dealing with decimal fractions is the same as dealing with ordinary integers through the positional representation, except that you have the added difficulty of keeping track of the decimal point. However, as noted above, this realisation is definitely not present with pupils, maybe not even among most adults.

No power of 10 can be divided by three, without remainder. This is easy to show mathematically, although it is not clear whether this is also apparent to pupils. After all a big number like say

$$1000000000000000000000000000000000000000000000$$

is awfully big, and God knows what will happen. For a mature mathematician (or any adult) there is an implicit assumption of infinity (we are considering *all* powers of 10) which we assume is not present in the concepts of a pupil.

There is now a slightly different way of considering decimal fractions, not as fractions with a fixed denominator (a certain power of ten), but as an expansion.

$$1 + 4 \times \frac{1}{10} + 1 \times \frac{1}{100} + 4 \times \frac{1}{1000} + \dots$$

This way of looking at things emphasizes the special position of the digits, and extends the analogy with representation of integers, when the position gives the (positive) power of ten. This is a slightly more cumbersome way than the first one I pointed out, whose simplicity may be startling and a bit disappointing. The advantage of this point of view is that we can now think of a ruler, say with the integral lengths marked off, then a refined division marking off (with less pronunced bars) the tenths, and then within that the hundreth. Practical considerations, soon limit the number of subdivisions, but the analogy should be clear to the pupils. The decimal representation, as suggested in the introductory passages of this essay, establishes a natural correspondence between numbers and length (of the continuum) as a sequence of

[8] Traditionally percentages refers to numbers between zero and one, thus often refering to probabilities, and only rarely do we encounter percentages above 100. I was startled when I first came across the notion of 150% refering to the length of the tail of a certain monkey compared to its length of the trunk, in Brehms 'Djurens Liv' at my grandmothers. I instantly understood what was meant, and was delighted by the extension of the notion.

[9] This notion I have almost never encountered in American usage, although of course most educated people would know, or instantly figure out, what is meant.

[10] I should not comment on this really, as I do not know to what extent floating numbers are hard-wired into computers or merely part of the soft-ware, only in the latter case, can I claim that my comment is relevant.

successive approximations. In fact there is established not only a correspondence, but an identification. Confronted with the fraction

$$\frac{886731088897}{627013566048}$$

we feel slightly uneasy, and would prefer to see it in decimal form, to get a handle on it.[11]

More to the point, this approach invites the extension to consider infinite expansions, i.e. an interminating sequence of decimals. One cannot very well consider the quotient of two infinite numbers say

$$\frac{141421356237\ldots}{100000000000\ldots}$$

as it is very hard, not to say impossible, to make sense of the numerator and denominator independantly.

To return to the pupils. They are confronted with the problem of writing 1/3 in decimal notation. They quickly find out that this is not possible to do with only two decimals 0.33 representing 33% as $3 \times 0.33 = 0.99$ Now it is not clear whether they try 0.333 next, or that they know, as remarked above, the standard procedure of dividing remainders, to produce a decimal expansion. The noteworthy thing is that they seem to have no qualms about writing down an infinite sequence

$$0.333333333333333333333333333333333333333333333333333333333333333\ldots$$

and assuming that this has a meaning. The corresponding notation

$$\frac{333333333333333333333333333333333333333333333333333333333333333\ldots}{1000000000000000000000000000000000000000000000000000000000000\ldots}$$

has no obvious meaning, as noted above. What does this mean? One interpretation would be that they have an intuitive understanding of the notion of a continuum. That in there attempts to mark off 1/3 on the ruler, they realise that they are always slightly off, and have to go down to finer and finer subdivisions, in a process that does not end. But that this process nevertheless correspond to a 'number' interpreted as a length. In mathematical jargong, that the real numbers are complete. They may also look upon the division algorithm as a kind of program which never ends, giving as an output a never terminating output of 3's. Most likely a combination of both is probably at play, where the second interpretation may be the one which is most explicit in their mind, while the first is the subconscious anchoring of mathematics to some kind of reality.

I do not believe that it takes some special talent to discover this, but that it is an insight almost all pupils make. I also believe that most pupils are intrigued by the phenomenon, but that its wider implications may be lost upon the majority. In the paper, this realisation is considered as a brush with the concept of infinity, and reading the paper one almost get the impression that the authors believe that this almost involves a personal discovery of infinity. I would like to devote the remainder of my essay to the notion of infinity.

**Infinity**

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[11] In fact the corresponding decimal fraction is the one 1.4142135.. displayed above. The reader may guess that these fractions are not chosen randomly, in fact they provide approximations to $\sqrt{2}$. The fraction above $(p/q)$ is a solution to Pells equation $p^2 - 2q^2 = 1$ found reqursively from the 'primitive' solution 3/2 through $p_{n+1} = p_n^2 + 2q_n^2 \quad q_{n+1} = 2p_nq_n$. We find from the equation that $(p/q)^2 - 2 = 1/q^2$ and thus $(p/q - \sqrt{2}) = 1/(q^2(p/q + \sqrt{2})) \leq 1/(q^2 2\sqrt{2})$ Given the size of the denominator $6.27.. \times 10^{11}$ we are talking about an error of the size of $10^{-24}$ thus the first 23 digits (after the decimal point) in the decimal expansion of the fraction are correct. This method gives a powerful way of computing literally millions of digits of $\sqrt{2}$. The implementation on a computer is however not straighforward, as the standard types of numbers are assigned very limited storage. Some simple soft-ware routines, dealing with long arrays of digits, need to be devised.

There are two notions of infinity, that are closely related, but which are nevertheless distinct. One synonyme for infinity may be 'unending'. The circle represents pictorially this notion. This is the idea of cyclicity, things repeating over and over again. To divide 1 by 3, using some kind of algorithm, involves doing the same thing over and over again. Always, always the same remainder - one, pops up, and you have to do the whole thing over again, as nothing have changed. There is a word in the language for that 'forever'. The process does not end. In programming language we have an unending (infinite) loop.[12]

Distinct from this realisation of 'unendingness' is to keep a record. While in cyclicity nothing changes, everything is just done over and over again, keeping the record means allowing things to pile up. You can look at the ever increasing string of 3's. They are unending, but not like the circle, they are taking off, usurping space.

Now do children have an intuitive notion of that kind of accumulating infinity, as opposed to repetition and forever? How many 3's are there? The pupil may write down five or six, and if she or he is obsessive, twenty or thirty 3's may be written down. But twenty is not a big number. Does the child realise that there is something different from those 3's popping up from the comfortable fact that after each sunset there is another sunrise, which the child will (always?) be lucky enough to experience? Infinity is very large indeed. Does the child envision all those 3's stringed out along the equator, circumscribing the Earth, small scribbled 3's, traversing the bottoms of oceans, climbing mountains, dipping into rainforests? In fact the 3's could be made to cover the entire surface of the earth, small 3's, written under each stone, to be found on each leaf of every tree, covering the surfaces of lakes, rising up as giant clouds, falling down as frozen precipitation, covering the Earth like growing glaciers. In fact the whole earth could be composed by 3's, wherever you dig, no matter how deeply, you encounter nothing but tiny 3's part of this unending accumulation. The 3's can be made to fill the space, raching to the sun, the nearest star, filling up the Milky Way. No end of 3's. Such a concept of infinity fills you with nausea.

In fact it is easy to consider the infinity of integers $1, 2, 3 \ldots$ after each integer there is another; but it is far more challenging to consider huge numbers. Astronomical numbers tend to awe people. Numbers with many zeroes. The number of mm's to the sun is of the order

$$150000000000000$$

The number of grains of sand, filling up a sphere encapsulating the orbit of Saturn[13] would be of the order of

$$12000000000000000000000000000000000000000000000000$$

Those are large numbers, and your hand gets tired writing down all the zeroes. But this is nevertheless easy work compared to ticking off each grain of sand by hand. Why tick off the zeroes one by one? Why not just note the number? Mathematicians do, and pupils would easily be persuaded to follow suite, once subjected to the initial drudgery. And we write

$$15 \times 10^{13}$$

and

$$12 \times 10^{48}$$

respectively. This is much less work. But just imagine that the exponents themselves are big numbers. How many different books can you write of 500 pages each, each page containing 50 lines, each line 50 characters, and with a choice of 70 different characters for each character? (Including upper and lower case, digits, punctation marks, spaces etc). It is easy to work it out. It will not be done, suffices it to say that the

[12] In the old days (thirty years ago) a nice illustration of this was to use an electrical counting machine and divide by zero. The machine was making a racket, as the cogwheels were turning. Obviously the machine was just subtracting zeroes, changing nothing. Unless you were willing to wait for the inexorable decay caused by erosion, you had no choice but to pull the plug.

[13] In a classical study by Archimedes - the Sand Reckoner, he tries to compute the number of grains of sand to fill up the 'imagined' Universe in his time, which would correspond roughly to the volume of the sphere just described. The challenge of Archimedes, trivial today, was a means of representing big numbers. The reader can find an English translation in 'The World of Mathematics' Vol I. page 420.

number itself would take almost two books of that size[14]. The number of digits would hence be equal to the number of characters in the two books, thus the exponent itself would need to be written in exponential form.

So we can start writing towers of exponentials.

$$10^{10}$$
$$10^{10^{10}}$$
$$10^{10^{10^{10}}}$$
$$\dots$$

Typography becomes a bit unwieldy. But that is trivial, we only reinvent new typography. Let us write $10^{10} = 10[2], 10^{10^{10}} = 10[3]\dots$. What about 10[1000000] pretty big number. It is very hard to get a handle upon it. What about 10[1000001] another pretty big number, probably bigger than the first? You bet! In fact we have

$$10[1000001] = 10^{10[1000000]}$$

This number is in fact inconceivably much bigger than the first. In fact we have never before in our lives (unless we have happened to play the present game before) encountered two numbers that differ so much in size. Not by a very, very, long shot.

So note that $10[n+1] = 10^{10[n]}$ we have here an example of a recursive formula, and the integers $10[n]$ provide a rapidly increasing series of the same. Even modest numbers $n$ give inconceivably huge numbers. Let us continue the game. This wild series is only the first example. Let us write $10[n][1] = 10[n]$ let us now define the series $10[n][m]$ recursively. Why not define $10[n][m+1] = 10[10[n][m]][m]$? What about such an innocous number as 10[2][2]? We have first to compute 10[2][1] which is a billion. Then we have to insert a billion, and consider 10[1000000000]. The reader is now either lost or somewhat jaded, and in the latter case he or she may think that this is no longer such a big deal. What about 10[3][3]? The reader may get the point. It is pointless to continue[15]. The moral is that big numbers can be very, very big. And the paradox is that the infinity of all numbers seems much less daunting, as it is just a matter of ticking them off, one by one, adding just another one. The exercise of this big numbers above shows that the simple method of ticking off one by one simply does not work. The vertigo this may have induced in the reader is also induced on a more modest scale, through the contemplation of astronomical distances. We are just a speck on the Earth, which itself is but a speck in the Solar system, which itself is just a speck among other billions of stars in the Milky Way, which by itself is just a speck among all billions of galaxies in the universe. We take a few recursive steps, and that is it, resulting in a huge astronomical number, but compared to our exercise above, it is really puny. After the Copernican revolution, and before the modern 20th century estimates of the size of the universe, the Galaxy of which we are a tiny member, was thought to be infinite, extending in all directions. This gave rise to various anomalies, Newton wondered why all the stars did not collapse by gravitation to a point, but obviously it could never decide what point, as all would be equal; and Olbers speculated why it was dark at night. In a more conceptual way it is hard to think of the Universe as limited, or time having a beginning or an end, because we always ask, what is beyond, and what happened before and after. Such psychological blocks may give rise to confirmation of Kants idea that the sense of space and time, as infinite and unending, is part of our basic inate concepts. Is thus infinity part of our inate concepts? We all have a notion of unending, but are we all familiar with the consequences of accumulation? I doubt it.

What about literature? Is infinity treated? Occasionally but not often. Let me restrict myself to two examples, which for some reason or other made an impression on me when I was young.

The Swedish writer Hjalmar Söderberg (hardly known outside Scandinavia) wrote a short novella -

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[14] In fact about $1.85 \sim \log 70$

[15] One may be inspired to have a competition. Every participant is given a small piece of paper, and a limited time, to produce the definition of a big number. The one with the highest number wins. Although it might not be so easy to determine. The definitions must be self-contained, wise-guys are not allowed to add one to the number of their neighbour, because he or she might also be a wise-guy.

Drömmen om Evigheten (The dream of eternity)[16] in which he ponders the consequences of an infinite existence. He imagines it to take the form of an unending succession of waking up into new dreams, each of them closer to reality than the previous one. The dream of an immortal soul is one of the most powerful supportive phantasies of mankind, and many of us take comfort in the hope that our existence will continue for ever and ever, and in endless bliss to boot. One wonders whether the proponents of such views have ever seriously thought about infinity. The protagonist of Söderbergs story does. The story was a product of the fashionable pessimism of the 'fin de siecle' (written in 1897), and the setting is the fashionable world of leisure and dissipating pleasure, spiced with philosophical reflections. 'It is horrible to be extinguished, but it is also horrible not be able to be extingushed' one of the character muses, while the other suggests that somebody should find the golden mean between the infinite and the finite, as represented by eternity and time, and thus provide the most attractive of all possible religions. The protagonist is saved from his dream by a night-mare. As he climbs the stairs to his abode in his apartment house, there always appears a new floor, just under that of his destination. It does not take too big a number to convince the dreamer of the futility of infinity.

The story is interesting so far as it does connect the idea of infinity, with the idea of immortality. Counting is a way of constantly evading death. There is always just one number ahead, just as the sun will rise next day too. As a depiction of infinity the attempts of the author are feeble. The protagonist quickly loses count. But that is a minor detail, the main point is that infinity, or as in the novella, the idea of eternity, may not be such a great thing after all.

A more imaginative representation of infinity, or at least of a very big number, is supplied by the much more well-known writer Jorge Luis Borges. In his short story *The Library of Babel* a giant library, containing all possible books, is envisioned. He is very meticolous in his presentation. The library consists of hexagonal cells, each identical, with the same number of shelves, the same number of uniforms books, each containg the same number of pages, lines per pages, and a total choice of twenty-five characters, including two punctation marks and space.[17] The narrator remarks that the total number of books is huge, but not infinite, but a powerful approximation thereof, as the totality of all hexagonal cells are said to form a sphere with no center and an inaccessible boundary. Librarians are drifting around in this huge library, chancing upon books, almost all jibberish, however, from time to time chancing upon scattered lines, which may make sense in some ancient or future language. The realisation though that all possible books are contained in the library is a source of joy, as all wisdom can be encoded in books. In particular the wisdom that summarizes all wisdom can be found. But also such mundane texts as the story of your life, translated in all possible languages. Any book you can think of, and more, are already part of the library. It is infinite in the sense that you cannot conceive of something that it does not contain. But now the mathematically attuned reader, like the author himself, thinks of self-referentiality. In addition to all possible books, there are also false catalogues of the library (the discerning reader here discovers a, probably unintentional slip, of the author), as well as the demonstrations of the fallacy of those catalogues, as well as the demonstration of the fallacy of the true catalogue (which the reader knows cannot exist). The search for those various texts of vindications turn out to be futile. The number of books, although strictly speaking not infinite, appears so to the feeble men, whose tenure in time is but brief. In desperation some people seek to purify the library of jibberish, but their misapplied zeal, although performed for centuries, makes little dent. Each book may be unique, but each one differs from millions of others, by the change of a single character or comma.

The fundamental point of the short story needs not concern us, namely that the mere physical existence of a text, a string of words, has no meaning, no matter how, wise, unless properly interpreted[18]. What is interesting is the 'Gestalt' of the almost infinite, remarkable for a literay work. However, a mathematician

[16] Hjalmar Söderberg (1869-1941), born and bred in Stockholm, known for short stories and short novels. I doubt whether this one, or any other of his stories, were ever translated into English. Some twenty odd years ago, when I first encountered it, I was moved to supply my own

[17] The exact numbers of pages, lines, etc are given in the text. Recall that the language is Spanish, so only twenty-two different letters are needed, although the library contains books in all possible languages

[18] If you could build chemicals, like the chemists do for illustrative purposes atom by atom using kits, there would be no chemistry. The point of a chemical is to aid in the process of building other chemicals. This is what is meant by the chemical property of a chemical. In a similar way books beget books, and ignoring the intermediate agents, readers and writers, this is how books come into existence, not through

could expand the metaphors, through various computations, which we will forego[19]. The concluding sentence of the story, speculates whether the library is not after all infinite, in the sense of being without boundary. Removing yourself far away to the periphery, the library will only repeat itself. A picture of a huge, but finite universe, complete in itself, with no boundary. In essence not too different from prevalent cosmological theories, except as to size.

Nietzsche believed in the doctrine of 'eternal recurrence'. Even if it did not play a major part of his philosophy[20] it was an integral part of it. I used to believe that he was inspired by Indian Mythology, but in fact he had a mathematical argument to uphold it, an argument that can be traced back to Heine. The gist of the argument was that the universe was finite and that there was only a finite number of material positions. Thus eventually we would have to start over again. Nietzsche was aware that the timespans must have been enormous, and hence that it was more of a meta-physical (or mathematical if you like) nature and of little immediate concern. It is not clear how Nietzsche thought (it seldom is, and that is, according to many, his main charm). It could either mean that everything recurs literally that time is cyclical, without end. However note that in absence of an independant recording, there is no way we can tell what orbit we are in, the very question becomes pointless, as all the orbitings are done so to speak 'simultaneously'. Or it could simply mean that every event must recur, but the events do not appear necessarily in the same order.

Let us make a small digression to illustrate the difference:

Consider an infinite decimal expansion. $0.333\ldots$ It is boring. Nothing happens. You could simply think of it as three occuring, over and over again. The decimal expansion itself might not even 'realize' that it is infinite. It only senses '3' and does not keep track of the '3's turning out. It would be an instance of cyclical time. Unending, maybe infinite, but only to an outside observer. A decimal expansion of any rational number is similarly repetitious, although the periods can be arbitrarily long[21]. The reason is that the same remainder will eventually be encountered again, and thus the same process will start again. Cyclical time, but somewhat more interesting.

Let us now take the decimal expansion of say $\sqrt{2}$. It is not a rational number, that is easy to show[22]that it is a number, is something else, to which we will return in the next section. There is a simple algorithm (discussed in one of the footnotes) that shows how you can produce its decimal expansion in principle indefinitely. However as the digits turn up, they do so unexpected. The same type of digits occur over and over again, after all there is only a choice of ten. But they do not occur

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random walks. For the modern reader one may also think of programs. A program that is merely written but not implemented on a computer, is inert. The point of a program is to affect the circuits of a computer, maybe producing a new program. In Biology we may think of genetic codes. To encode your genetic data, does not mean that you have been duplicated, and hence providing a safeguard from mortality. A genetic code needs to interact (chemically) with the environment, by itself it is dead

[19] Borges is not a mathematician, this is what makes the story especially intriguing to a mathematician. Borges is content, like in all his short stories, to merely suggest. A mathematician could easily expand the story, by e.g. computing how rare indeed a non-jibberish work would be

[20] And was somewhat cheaply refuted by one of his biographers - Kaufmann, using two wheels rotating at incommensurable speeds. A more direct refutation, in the same spirit, would be a simply counting of the integers one by one.

[21] As an example I have included as an appendix the first period of the decimal expansion of $\frac{1}{65537}$ to see how complicated things get even for relatively small denominators. A simple program, and the computer nowadays spits it out essentially instantaneously. It is one thing to know in principle that rational decimal expansions are periodic, quite another thing to actually behold a physical instance.

[22] The standard argument uses so called unique factorization of integers (all we need actually is that squares of odd numbers are odd) and proceeds by simple contradiction. If $\frac{p}{q} = \sqrt{2}$ (where $p/q$ can be assumed to be reduced) then $p^2 = 2q^2$. Thus $p^2$ is an even square thus $p = 2t$ is even. Hence $4t^2 = 2q^2$ thus $2t^2 = q^2$ hence $q^2$ is an even square, and thus $q$ is an even number, contradicting our assumption of no common factors. Alternatively you would be able to shave off factors of two from both $p$ and $q$ indefinitely. Yet another instance of infinite thinking. A similar proof that it cannot be written in a finite decimal fraction might be easier to grasp, especially for a pupil. Look at its last digit. We can assume it is not zero. As its square has to be zero, we have a contradiction, or if you prefer, the last digit has to be zero, and hence the next to last etc all the way until the integral part. By using other bases than ten, this proof can be made to work in general. The technical machinery for this proof is very limited, but of course somewhat sophisticated. The ancient Greeks discovered it, and it had far-reaching philosophical consequences as to the nature of the continuum.

in the same order. It is reasonable to assume, although hard, maybe even impossible to prove, that any combination of digits will eventually occur.[23]Not any combination of fixed length, but of arbitrary length. Thus the production of the digits will in a sense give you a Super Borgesian Library of Babel. In particular all 'books' are present, suitably decoded. (It does not matter how). Take your favourite play by Shakespear (always this Shakespear). Given a convention of decoding, you can find it somewhere along the expansion, and in fact infinitely often, with minor variations, and translations, both masterly and abysmal in any kind of conceivable language. Somehow you get the impression that the innocous number of $\sqrt{2}$ which contains so little information, in fact buries in itself all possible information, and thus preserves within itself the whole world. The Library of Babel is a fiction, but the expansion of $\sqrt{2}$ is not a fiction, it exists, or? The physical representation of the Library of Babel is a figment of the imagination[24], but surely in some sense all books exist even if they are not printed out, just as all integers exist, even if no one has bothered to write down their (decimal) expansions, just as no one has ever bothered to write all books[25] . Because a huge number contains as many digits as the characters of a book, and some of them may suitably interpreted be read as books. But in order to enjoy this passage, you need to know where to look. In fact chances are overwhelming that its first occurence will appear at the $N$th digit, where $N$ is such a large number that its decimal expansion is as long as the text itself. Thus the instructions are as complicated as the very text itself. The adress so to speak simply constitutes a tortous decoding of the desired text. The information hidden in the decimal expansion is useless, as you need as much a priori information as the information you are seeking. There is a mystery to the unending sequence of digits appearing in a decimal expansion, but they carry no meaning, disclosing no secret messages, although it is tempting to believe so.

Now the actual infinity is a large entity indeed. Does it exist in a physical sense, or is it just a formal idealistic chimera born out of the feverish human brain? What sense do those big numbers make that we discussed above? Are they just formal games? It is impossible to produce physical things to match up with those huge numbers. All the particles in the known universe would not make a dent. But you can also count things that you cannot lie your hand on, like all possible books, or all libraries, meaning collections of books etc.

Big numbers involve much more of a challenge to your imagination than infinity does. Nevertheless it is something that fairly small children can enjoy. When it comes to infinity, one expects the notion of it to occur even earlier, maybe at the age of three or four[26] This is definitely something that might be worth pursuing.

Can we count infinity? If we try we will encounter paradoxes. Already Galileo noted that there are as many even numbers as there are numbers. The proper part can be equal to the whole. This was illustrated by Hilbert, in an effort of popularization. The Hilbert hotel has an infinite number of rooms, and each one is

[23] This statement probably holds for almost any non-rational number not defined via its decimal expansion. In particular for standar numbers as $\pi, e$ etc. The point is that those numbers are just harder to compute, and their decimal expansion may be more 'mysterious' whatever that may mean.

[24] A virtual Library of Babel does exist. How do you find a book. By its title and author in ordinary case. But there are too many books in this library, there simply are not enough titles with authors to go around. You can only write so much on the spine of a book. Thus the title becomes identical with the contents. How are the books stored? Alphabetically! Hence the first question the librarian asks you is what is the first letter. If there were doors for each initial first letter, you would walk through one of them, and then be confronted with a second choice of doors. In our Euclidean Universe you would have to walk longer and longer, as the distances between the doors would increase exponentially. (In the Hyperbolic Universe you could avoid this, but you would nevertheless be forced to walk through as many doors, as there are characters in the book) But the librarian saves you the trouble of walking, she only asks you to give her your choices. She types them down. After a few days, the book has been typed. Maybe someone binds it for you, and hands it over with a flourish. Here it is. All books are to be found in this virtual library. Finding a book is equivalent to writing it!

[25] What I remember most vividly from our music lessons at school, was that Mozart (unlike Beethoven) wrote down his scores without effort. According to the story, he never made mistakes, or changes; but only remarked that all Music exists, it is just a matter of copying it down

[26] I recall at a tender age seeing a can of paint illustrated by a little girl holding up the same can of paint, realising that this must go on indefinitely. This is in no way an unusual observation I believe. Also the mutual reflections of mirrors, create the illusion of infinity, as well as implying it theoretically. In fact the thought experiment of a perfect cube, formed by perfect mirrors, would create an Olberian universe of stars, if a single source of light would be inserted in it. The way out of the Olberian paradox (of infinite light) is to conclude that the speed of light is finite, hence the reflections do not form an infinite universe, only an expanding one

occupied. Come a new guest. No problem, each guest simply moves to the next room up. Come a bus-load with an infinite number of guests. No problem, each guest just doubles his room number and moves to the corresponding room leaving all the odd-numbered vacant. What about an infinite number of buses, each with an infinite number of guests?[27]. With infinity there is really room for waste.[28]

The fact that you cannot really get a tactile sense of infinity. An infinitude of money and earthly riches[29] would need (as well as be required by) an infinitude of time to be enjoyed. The horrors of such a prospect are not unfamiliar to the protagonists of Söderbergs short novella. On the other hand any finite number involves an end, which by itself is unacceptable. Where indeed do you find the Golden mean between the terminating and the interminable?

Cantor approached Infinity head on. Counting an infinite number of objects, means putting them into a so called one-to-one correspondence with the natural integeras. This is an extremly simple idea, familiar to most children, as this is what is meant by counting. You pick out the (discrete) objects one by one, as you do so you go up the ladder of numbers. The practical problem is to keep track of the objects you have already counted, so you do not count the same object twice, and also so you know when you are finished.[30] The last number you reach, is the number of objects. Of course if there are very many objects, (like the books in the Borges' Library of Babel) this stratagem does not work in practice. Then you have to design a more clever way of setting up the correspondence, and also invent a new terminology for numbers, as you quickly run out of the everyday words like thousands, millions, billions and trillions...

But if you keep on forever, do you ever get finished? And here we make the giant leap. We imagine so. Somehow we need to be able to describe this 1-1 correspondence in a short succinct way. There are as many even numbers as there are numbers. $n \leftrightarrow 2n$ Note we cannot just tick off the even numbers one by one in practice, we take the leap, and 'imagine' this being done forever. And this 'imaging' is mathematically expressed. If we want to count an infinite number of buses each with an infinite number of passenger, we need a slightly more elabourate mathematical 'imagining'. One example would be $(m,n) \leftrightarrow (2m+1)2^n$. Is this magic, or is it legitimate? We cannot, even in principle, affect this 1-1 correspondence in any physical way. What gives it mathematical legitimacy is the rule, expressed in a few symbols, that gives a kind of 'automation' to the process. In each case we know what to do, in fact the rules would allow a machine to do it for you. The key is that although the process is infinite, it can be decoded in a finite way. We can convey the necessary information to complete the task in a finite set of symbols. It would have been different had we done this coupling in an *ad hoc* manner.

Mathematics is essentially a way of saying things about infinite objects in a finite way, in other words to find the 'essence' of an infinite process. The infinite object of study *par excellence* is that of the integers. We somehow believe in them, although we may not believe in an infinite number of physical objects. The belief is almost tangible. We can feel them being ticked off one by one, although this is impossible in practice. (The construction of very large numbers should cure us from our naivity in that regard). If we look (heavens forbid) at mathematics as a formal game of symbols, our reflection on this game is an appreciation of infinity, as an emergent feature. The 'existence' of infinity is hence a philosophical question. Believers in the mathematical infinity, like the set of all natural numbers, are sometimes dismissed as Platonists. Most mathematicians are Platonists, although some play the game of pretending not being Platonists. The

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[27] As a child I used to amuse myself with taking a number and doubling it, and doubling it again. In this way I got the beginning of an infinite series of numbers. At some time I realised that you would get different series by starting with different odd numbers, thus decomposing all the integers into an infinite number of infinite sets, although I did not realise the 'significance' of this discovery until I was an adult. Incidentally this remark gives a solution to the problem

[28] Politicians are accused of a short time perspective, but nevertheless the future is seen as infinite, and thus you can bank on it indefinitely. Thus the desire for sustained exponential growth. In fact the closest we have been to sustained exponential growth in history, at least for limited times, and thus providing the best everyday approximation to infinity, have been the phenomena of Hyper-Inflation, like in Germany in the early twenties.

[29] Mathematicians are rightly proud of their Infinities, but of course they cannot cash in on it. One may truly in this regard claim, like Christ, that their realm is not of this world.

[30] This is why it is so difficult to count sheep, even if they are not so many, and explains why this keeps you awake at night.

question of the independant 'existence' of the whole entity of the integers, i.e. believing in an *actual* infinity as opposed to a *potential* is akin to the question of the 'existence' of God, or the 'existence' of independant 'conscioua', thus transcending logical inquiry.

Imagine an infinite test. For all triplets of integers $(x, y, z, n)$ with $n > 2$ and $xyz \neq 0$ (to avoid trivial examples) compute $z^n - (x^n + y^n)$ until you get 0. This is a simple, if somewhat tedious task, and can in principle be given to a (n expanding) computer. Two things can happen, either we get a zero, or we never get a zero. If we get a zero we have found a counter example to Fermat. The counter example would take the form of a calculation, most likely a horrendous calculation, involving numbers much bigger than those we have considered so far. But it would be a finite calculation, never mind that the number of operations may vastly exceed the number of events that has ever occured in the Universe since the Big Bang. Say that it never stops? How would we find out? But if it never stops then it must be TRUE. I write true with capital letters, as this is part of a judgement and an assigning of meaning to 'true', transcending mere convention. Do you believe in this? If not, what meaning should we assign otherwise to Fermats Conjecture? Now, as it happens, there is a proof of Fermat, a finite number of symbols, that in principle allows us to 'understand' why it is true. A fact that in principle could have been checked by an infinite number of tests, is reduced to a finite statement, - the 'essence' of why it should be true, relieving us of the literally infinite drudgery of checking.[31]

We have seen that infinite sets are big. And in particular the simplest and most basic - the integers. So big that small parts of the same can be equivalent to it. To assume the existence of such a huge set really gives us a huge advantage over 'mere machines'. Those may be better at handling big numbers and perform tasks to obsessive lengths, but 'they' fail to be able to reflect on the nature of infinity. No wonder why we are superior to machines and formal algorithms, as we are stacking our cards so much against them.

The, so far at least, ultimate statement on the bigness of the Infinity is given by Gödel. The integers constitute a giant Library of Babel, in fact the different numbers can be thought of as books,(formulas, programs, proofs) and in particular say pertinent things about themselves. There is no need to go into a proof of Gödels theorem at this stage, although the reader may by now have appreciated many of the ingredients that go into it. The crucial one is the constructive use of self-referentiality known as the diagonal principle. (Detractors of Mathematical Logic may claim that this is the only trick in the business, but with such a trick, who needs more?). We will return to it in the last section.

Let us conclude by refering to the mathematical physicist Penrose, who in a popular acclaimed book 'The Emperors New Mind' set out to debunk Artificial Intelligence. The pivot of his attack was in fact furnished by 'Gödels theorem' which asserts the transcendancy of meaning above formality, and in particular shows that creativity cannot be programmed. The technical content, however, pertained to showing that there are 'truths' that cannot be proved in any logical system powerful enough to pontificate on the integers. The so called 'incompletness theorem'.[32]. As Penrose allows himself the power of Infinity, there is, as noted above, no surprise that he wins againt a foe, who is denied its assistance. The proverbial Man on the Street would throw up his hands in exasperation if it was suggested that he could be emulated by a machine, a formal algorithm. He would invoke his sense of humor, the depth of his feelings, and the existence of his soul. Penrose does the same, only that he is more circumspect in his language. It is not that I do not sympathise with his quest, on the contrary, I also believe in Infinity.[33]

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[31] One may wonder whether there actually are 'truths' which can only be checked painstakingly case by case, and never amendable to finite reduction, and thus expressed in logival proofs. In fact there are, and we will return to this later.

[32] Penrose spends quite a lot of time on elucidating the logic behind Gödel in the language of programs a la Turing. He also dabbles into a speculative notion of quantum calculators to try to explain the wonderful unpredicatbility of the human mind. In the early 80's Hofstader became a kind of 'guru' by the popular success of 'Gödel, Escher, Bach' in which he in a leisurely way treads the same ground as Penrose, but without the ulterior motives of the latter

[33] Do all Integers exist? If not point at one, that does not! In fact the very thought of an integer brings it into existence. Or maybe drags it up from our collective unconsciousness.

To see the world in a grain of sand
and heaven in a wild flower
*to hold infinity in the palm of your hand*
*and eternity in an hour*

Those often quoted lines of Blake[34](italics mine) may be used to carry much more weight than initially intended by their author. Their point is, however, obvious, (even to the author?)

So far we have thought of infinity as something big, huge numbers as phantoms of the imagination, inaccessible, except through the play of mathematics. Can you *hold infinity in the palm of your hand?* Let us look at our old friend again.

$$0.3333333333333333333333333333333\ldots$$

This is an infinite sum. Can an infinite sum be finite?[35] People must have wondered about this in history, at least so did the Greeks, or at least Zeno. The story of the Hare and the Tortoise is too wellknown to be recalled here[36] What is the paradox? An infinite sequence of events are described, and they all seem to take place in finite time. Does it not take for ever to count to infinity, ticking things off one at the time. Can you really count to infinity in a few seconds? So while we have been gazing towards the heavens for the ever receeding infinity, it has all the time been dwelling in front of our eyes. The number of points along the ruler, are like the stars, infinite. But as we will eventually understand, another kind of infinity. To identify a point we have too look closer and closer, magnifying more and more, as we focus on an ever smaller interval of scrutiny. The process is infinite. To pinpoint the number 1/3, by means of decimal rulers, we need to go through all levels.

The decimal expansion of a number is a kind of address. It tells you exactly where to look for the number. But it takes for ever to find it, in fact you will never reach it, as the sequence of steps is infinite. But will there be something beyond all that counting? Will there be a number, a point, waiting for you? Or are the points in the palm of your hand as inaccessible as the stars in the sky, placed on an infinitely distant firmament. From a physical point of view it is. The atoms we are made of, seem as distant and inaccessible as the objects of the sky. How do you identify an atom, how do you point at it? And mathematically the size of an atom is not a very demanding task in representation. Just as we have unbelievably big numbers, we will also have unbelievably small, just by turning things upside down.

Our lives are finite in length, but surely each moment in time is a point, and there is an infinite number of points. Thus are not our lives infinite after all, each moment can carry as much weight as a lifetime. If we just look close enough. In fact at the end of Borges story he adds a postscript in the form of an editors note. The whole library of Babel can be compressed in an arbitrarily thin book, provided the pages are infinitely thin. Borges (or his imaginary editor) imagines each page being able to split in two. Now Borges does not seem to be aware of that this is another kind of infinity, that the pages of his thin book, do not correspond to our usual notion of infinity, but an even deeper one - the infinity of the continuum.

Do we all have the same inate idea about the continuum? It can be described mathematically though axioms. Such axioms would probably confuse most non-mathematical adults, to say nothing about pupils. But the essential feature those axioms aims to formalize, are, I believe accessible to us even at an early age. Yet the notion of the continuum is not unproblematic. Is it 'choppy' or 'smooth'? Psychologists tell us that

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[34] They consitute the initial four lines of 'Auguries of Innocence' set off from the rest of the poem. Few lines, if any, have so often been invoked in scientific presentations, which should perhaps elicit an apology from me. Readers with a perfect recall, or who check the source, may note a slight modification of the original, which may be due to creative recall or erosion of memory, but in any case confirms a personal claim.

[35] Do pupils wonder about that? I recall being intrigued that a sum of infinitely many terms could actually be finite. Such a reaction presupposes that you look at the sum anew, not as a result of process (of long division) but as an entity on its own.

[36] But we should not forget that to each new generation of pupils it is fresh as dew. A scientists may be impatient with the philosophers, who still worry about it, as its resolution seems beyond any reasonable and fruitful doubts. But maybe philosphers are as fresh in thought as innocent children.

we can only process about forty bits of informations per second, this would imply that we experience time in chunks of about 1/40 th of a seconds duration. So those are the 'moments' of which our lives are made. Thus, discounting lack of consciousness during deep sleep, the average life is given about $10^{11}$ moments, and that is all.[37] This is a far cry from the infinity of moments we spoke about above. However, subconciously we receive several millions of bits of information per second, most of which is processed by our visual system, making sense of the world. One also may surmise that many other activities are processed subconsciously at great speed, trivial examples may be the feats of calculating prodigies (who with few exceptions seem to be autistic). The great activity of our subconscious, close to a million times as active, in terms of bites, may be a comfort, giving to our lives a depth, it otherwise would seem to lack, reassuring us that those brief moments do themselves have extension. Going further into Modern Physics, we encounter the phenomena of Quanta, and especially the unit of Plancks time, about $10^{-40}$ s[38]. Naively, at least, this would imply that time is after all a succession of moments, stringed out like the beads on a string of pearls. We are then back to the speculations of Zeno, when we go from moment to moment, but with the twist, that between each two moments, there is always another moments, as rational entitites all do exist.

Rational entities, or better still commensurable lengths, constitute a notion going back a long time. The discovery of Pythagoras, was really that the length of the diagonal in a square is not commensurable with the length of a side, was a blow to the notion that all 'numbers were rational'. There is hence more to the world, than there is to be found in your books.[39] Indeed there are more numbers than can be listed in books, as we will see shortly.

Do infinite sequences exist? Let us once again return to the infinite sequence

$$0.33333333333333333333333333333333333333333333\dots$$

Pupils have no problem with accepting it. It can be written down in principle. A meaning can be attached (1/3 which is actually much simpler than any approximation by decimal fraction). What about the periodic expansion of 1/65537? It looks daunting, but after a few pages of finely printed digits, it just repeats itself. There is no difference in principle between the single digit '3' and a few pages of 'random digits'. What about the decimal expansion of $\sqrt{2}$. It is infinite, and in a non-repeating manner. Most likely it is kind of random, and there is no 'formula' how to express the $n^{th}$ digit. Thus we cannot 'see' the entire expansion, like we can in principle 'see' the expansion of a rational number, or some other specially designed number[40]. However the statement that 'there is no formula' that allows us to compute the $n^{th}$ digit should be taken with a grain of salt. In an earlier footnote I have described an efficient algorith for doing so, but nevertheless I venture that we will never be able to determine digit number 10[3][3] in the decimal expansion of $\sqrt{2}$ (although the corresponding problem is trivial for 1/3 and I expect doable, by modular arithmetic for 1/65537).

We come now to an important point. There are expansions that allows us, at least in principle, to compute every digit in the expansion. There is some kind of formula, or procedure, or algorithm, if you prefer, that takes care of this. Somehow those numbers are predetermined. But what about a truly random sequence of digits, generated by say an interminable number of throws of dice? Now we are really taking a leap and imagining random sequences, undetermined, unpredictable, which can only be defined by the tedious, and interminably so, process of writing down digits in an ad hoc manner? We have previously expressed some doubt as to the real existence of the kind of monstrously big numbers we earlier have enjoyed constructing. But this is definitely something totally different? The big numbers were defined by some finite,

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37 It is hard to get an instantaneous grasp of such numbers, as due to our limited visual resolution. However brightness gives us a way of representation. The magnitude of the Sun is about -27, a bright star about 0. The difference in luminosity is about $10^{11}$ Assuming that the star is intrinsically about as bright as the sun, what we are seeing is actually the difference in apparant size of the discs of the sun and the star. The angular extension of a star is too small for us to notice directly, in fact too small even to be resolved by the most powerful telescopes. The great fallacy, of this I believe original simile, is that physiologically (Webers law) we tend to experience stimulation logarithmically rather than linearly. The classical classification of stars into magnitudes is an excellent illustration of this.

38 Extremly small with the measures of everyday, but mathematically trivial

39 An imperfect rendering from Voltaires Candide

40 Liouvilles number $L = \sum_{n=1}^{\infty} 10^{-n!}$ comes to mind

and in fact, very compact notation. Bad poetry usually requires more characters than 10[3][3][41]. But now we are describing infinite processes, which cannot in any way be finitely generated. We are really taking a huge leap of faith.

But let us with Cantor go straight on. Not only do all the integers exist, although we can only write down a truly miniscule fraction of them, but also all possible infinite sequences of digits, none of which we could write down physically, and only a tiny number of which we could give a formula for, and thus at least formally assert their existence.

Those sequences of digits will be identified with the real numbers. What it really says is that no matter how you wiggle down the scales in a kind of random drunken walk, there will at the end be something there. This is the essence of the idea of the 'continuum'. It cannot be proved, it has to be taken on faith, and in mathematical discourse taken as an 'axiom'.[42]. If we grant that, what happens? Can we count the real numbers, can we remove them one by one, so at the end there is nothing left? We can do so by the integers, ticking them off, at least in our imagination. If we are a bit more clever and technically adroit, we can do the same thing for the rationals, or more generally algebraic numbers (satisfying polynomial equations with integral coefficients) like $\sqrt{2}$. And in fact if we consider numbers whose decimal expansions can be calculated by finite formulas, i.e. sequences which are deterministic in some sense, or as the logicians would call them - recursively definable, they would still be countable in the sense of being able to ticked off one by one and eventually depleted. This takes cares of numbers like $\pi, e$ and Liouvilles number ($L$). In fact it takes care of any conceivable number you can think of, because the definition is so wide that anything not covered by it is 'inconceivable'. Does this take care of all the numbers? It depends if we grant the existence of 'inconceivable' numbers. Numbers who are so erratic that you can never in advance predict their expansions, they have to be written down, every single digit.

How do we find such a 'inconceivable' number, when we by definition is not even allowed to 'conceive' of it? So assume that we have ticked off all the numbers, i.e. numbered them by the integers. This is a grand assumption, it means that we have a long list, and infinitely long list, in which not only the number of items is infinite, but also the items themselves are infinite. We have this long list, a feat of imagination literally infinitely more daring than the pedestrian one of Borges. Now we do something even more daring. We read this list, number by number. We do not read all the digits of the decimal expansions, after all we are in a hurry, and we do not want to get stalled at the very first item. We also have brought with us a long tape, infinitely long by the way, but this does not bother us anymore, once we have gotten used to infinite chaotic decimal expansions, and a huge list of them all, this is now trivial. And what do we do? At item number $n$ we look at digit number $n$ and write down a different digit on our tape. So we continue. For ever. In the end what do we have? A long sequence of numbers, in fact a candidate for the list. But it is nowhere in the

[41] One should remark that the definition of the terminology should be part of the description, which would increase the number of characters needed, on the other hand poetry, even bad, requires a huge baggage of contextual knowledge (the meaning of the words to start out with) to make sense

[42] Axioms have two meanings, distinguished by the ancient Greeks, who talked about axioms and postulates. On one hand we think of an Axiom as something evidently true, and consequently so fundamental that we cannot explain it, or support it, with truths even more evident and fundamental. This notion of an Axiom ties in with the very epistomology of truth, supplying the pillars on whose support our entire worldview depends. Whether the Axiom is true or not, is a question of faith and also morality, and we turn to the Logicians for guidance and reassurance, like we in former times may have appealed to priests and shamahans. On the other hand we can think of it as purely formal, just a convention defining the game. We do not seriously ask ourselves whether the 'axioms' of a group are true or not, they define the notion of group; just as we do not ponder whether the rules of chess are valid. Axioms and what they signify are then devoid of meaning, this is the essence of the program initiated by Hilbert at the turn of the last century. However we can ask meaningful questions as to consistency of the rules of the game. When it is no longer possible to make a legal move in chess, the game is over. As to the Parallel Postulate, the Greeks did not consider it as evidently evident, and in fact unlike the other postulates of Euclid it involves the notion of infinity, what happens far, far away? In a remarkable foresight they decided to append it as an axiom. The formal character of this axiom became only apparent two thousand years later, with the birth of Non-Euclidean Geometry, which philosophically challenged our asymptotic notions of space. But this is another story, for another essay.

list? We have found a missing number, never mind that it took us an infinite effort. We can look back upon it, reflect upon it, and the conclusion is that the list was defective. So big deal just add the missing number to the top of the list. Another 'fella' comes along and does the same trick, even this list is defective. But we just add this and continue the process, infinitely many times if needed. But you are missing the point, is there such a list or not? We are not trying to construct it, this is after all not our job, we are simply asking the proponents to produce a list. If there claim is to be true, there must be such a list, and we have shown that the list is defective.

What about the missing number. In what way is it inconceivable? It took an infinite effort to write it down, we had to read through a whole infinite list. We cannot tell in advance what it will be, we cannot go through a finite number of items, and figure out the 'idea'. This construction is really a feat of imagination. In what sense does it makes sense, in what sense is this real?

Some mathematicians have qualms. Imagine that we only consider numbers given by finite formulas, they can fit into the list. In fact the list can be finitely described, or can it? In fact if it could, there would be a formula, in principle, for the $n^{th}$ digit, and thus the procedure could be automized, and rather than imaginging the thought experiment of the infinite list, we can retire to our chamber, scribble on some paper, and produce a recursively defined expansion, that is missing from the list. Thus in this, so called 'constructive' case, the list does exist, but you must write it down, line by line, and never be able to resort to the magic world of etc...

But once you have given Infinity your small finger, it gobbles up the rest. If we have accepted the reality of the continuum, and hence the second order of Infinity, which is the wonderful discovery of Cantor. There really is no stopping us. Where do we go next? Consider all the subsets of the continuum. Can that be set in a 1-1 correspondence with the Continuum? More generally taken any set, and what this is, is up to your imagination. What about the set of all its subsets. The Diagonal trick comes in handy again. If there is such a correspondence we can construct a funny subset call it Cantor for the sake of the argument, namely consisting of those members that do not belong to their corresponding subsets. In the putative 1-1 correspondence, there will be some element corresponding to Cantor. Will it belong to Cantor or not? In any case we get a contradiction, thus the set of all subsets has a higher order of Infinity, or more technically a higher so called Cardinality. So take the integers. The set of all subsets of the integers can be identified with the continuum.[43] Then we can take the set of all subsets of the continuum, and continue indefinitely. To each integer there corresponds an order of Cardinality. But we are not finished yet, not by a long shot. Take the union of all those sets. This is bigger than any of them. An infinity of infinities. Now continue, beyond all bounds. If the thought of large numbers has boggled the brain, what about this cascades of Infinities? But is there not a bound? The set of Everything. This surely must be the biggest set around. It contains, by fiat, everything. So unlike the integers, there is a biggest Infinite Cardinality? But what about the set of its subsets, it is already included, on the other hand it has bigger Cardinality.

The work of Mathematicians consists in building imaginary worlds, given certain assumptions. All ramifications are explored. And the world becomes more and more elabourate. However in most cases a contradiction will be encountered. This is like a Nuclear catastrophe. The whole world is destroyed in one go. And out of its ashes a single truth can be ferreted out, namely that the presumed assumption is wrong. This is known as proof by contradiction, but this prosaic description does not make justice to the elabourate activity, tongue in cheek.

So what is wrong. Let us go back to the proof of Cantor (which is but a variation of the Diagonal trick) and we encounter the set Cantor. It was, however, Russel who popularized this set around the turn of the last century, and the Paradox became known as the Russel Paradox.

What to do? Local contradictions, mathematicians can live with, in fact that is how they make their living. But a global contradiction! The Paradise of infinities of infinities, that Cantor had shown to the mathematician, was not something they, in the words of Hilbert, wanted to be expelled from.

How the Paradox was eventually resolved, need not concern us here[44]. Suffices it to say that one philosophical sect of mathematicians (known as constructivists) renounced the abstract notion of 'existence'

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[43] The correspondence is slightly technical, but can be elegantly expressed through dyadic expansions (i.e. using two instead of ten)

[44] What is a subset? defined by some 'property'! One has to be careful with what is emant by 'property'. An elabourate Axiomatics of Sets was built up in order both to avoid contradictions, and thus ensuring

which lies at the very foundation of this Paradise, and instead propagated for explicit constructions for any object of study[45]. For the working mathematician, philosophical niceties, are of minor relevance, most of them, with the exception of logicians, need not to dwell in the rarified atmopshere of large cardinals, those of the integers and the continuum, are enough[46].

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intrinsic consistency, by cutting off too wild flights of fancy, and also to encompass as large as possible part of classical imaginative mathematics. Axioms become then less of evident truth, than kind of logical contingencies. The late Gödel himself propagated the old ideology, that axioms should be natural and self-evident, and that the present inadequacy of set-theory (say the independance of the so called Continuum Hypothesis (one cannot write footnotes to footnotes so no explanation is provided)) was due to a lack of natural axioms. Axioms that should exist, but which so far no one had yet been able to formulate

[45] With the rise of computers, the demands of explicit, finite constructions, as opposed to general imaginativearguments, have made the constructive point of view imperative. Not surprisingly, many constructivistically minded mathematicians, have found their niche in computer science.

[46] The so called uncountability of the continuum is fundamental to every analyst, because without it modern measure theory would not be possible.