'The Evident and the Obvious'

Comments and Reflections on a paper

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Introductory Remarks

What constitute mathematical thought? This is of course a fundamental and general question, towards whose elucidation the field of mathematical didactics ultimately aims for, be it just for disinterested understanding or for the application to instruction. The purpose of this essay is two-fold. First the general one of dwelling on the question, to turn it around, and to reformulate it in various ways. Secondly the purpose is more narrowly constricted to comment on an attempt of given it a more focused scientific presentation, as exemplified by Mats Martinsson's (henceforth refered to as the author) **Det Evidenta och det Uppenbara** *studier av matematiskt tänkande med utgångspunkt i ett geometriskt exempel* (refered to as EU from now on). Much of the essay can be read without explicit reference to that paper, although some of it would be unintelligible, without resource to it.

'Mathematical Creation'

The title refers to the famous essay by Henri Poincaré, in which he dwells on the mysteries of mathematical creation. As this is also the point of departure of EU, it is very appropriate to start with a discussion of an essay, intended for a large audience, enjoying a wide appreciation and respect.

In the short essay (pertinent passages quoted in EU) Poincaré expresses puzzlement that mathematical deduction, which ostensibly proceeds by short steps of reasoning along universally accepted principles, is so hard for the general public. That this should be the case with mathematical creation is one thing, that is much easier to understand; but when all the steps are laid out, and there are no demands on active involvement, this is quite another.

One may naively think that the secret of mathematical mastery is a very good memory (keeping track of the meaning of many evolving concepts) and an ability to pay attention (in never slipping); but that assumption is belied by experience. Poincaré chooses to make the comparisons with skill in games, like 'whist' and 'chess', in which, superficially at least, the same demands are made in the pursuit of long chains of reasoning, keeping them separate and intact in the mind. Poincaré notes that he is a poor chess-player, and conversely that most good chess-players show no mathematical talent whatsoever.

Why is Poincarés memory so faulty when it comes to chess, but apparently flawless in mathematics? The first is obviously the case, Poincaré happily admits, but to claim the latter is to miss the point.

After having described some personal experiences of his own, all of them leading to the sudden illuminating insight, after a longer or shorter period of frustration, Poincaré is ready to formulate a theory.

One should remember that Poincaré writes an essay, not a scientific treaty; that he uses no systematic methods, save that of introspection and rethorics. He certainly did not have the ambition to say something definite on the subject, in fact he may have nothing more in mind that to amuse an audience and to fulfill an obligation. This does not mean that the major thrust of his conclusion, the mystery of creation, as it takes place in the subconscious mind, is not dear to him, as it is to countless mathematicians, who in his essay find welcome reconfirmation.

To do mathematics is to choose the right and fruitful combinations of ideas. But the number of possible combinations being so large, it is not possible to make them all up, and then inspect them one by one, rejecting all but the good ones. There must be some guiding principle, and here he evokes the sense of beauty. It is the sense of beauty that frees him from the demands of memory, arguments fall just into place (and he notes that the order of the arguments is more important than the arguments themselves). He also notes that it is only the illuminated insights that are provided by the subconscious, never the routines of a calculations.[1]

The problem with Poincaré is not so much that what he says is not true, but on the contrary, that it is irrefutable. It is also very general, he offers nothing, or very little, that distinguished mathematical thought from other kinds of thoughts. The work of the subconscious, and the sudden flash of insight, is not only familar to the advanced mathematician, struggling with an obtuse mathematical problem; but maybe even more to the man in the street, suddenly realising what he has been missing all along, admittedly concerning something far more mundane, but nevertheless equally compelling. One may speculate that the main reason Poincaré brought this autobiographical sketches to the attention of his listeners (and future readers), was not so much to pinpoint the special nature of mathematics, but to connect with the rest of mankind, to show that mathematical thinking is a human activity like any other; and thus most of all, to emphasize that mathematical thinking is not a mechanical activity, guided by rigid rules, but as creative as any of the arts.

It is not easy to take Poincarés essay as a point of departure for a scientifically methodical inquiry into the phenomenon of mathematical thinking, the essay is short on specifics, and generous on generalities. But EU nevertheless tries.

The Obvious as opposed to the Evident

The crucial point of EU is to make a distinction between the 'Obvious' and the 'Evident'. In everyday speech, apart from local idiosyncracies, the two words are used interchangeably; but here, for the sake of argument, their meanings are focused, to serve as formal (if suggestive) labels for two fundamental concepts. It is in the discovery and elucidation of those two concepts, that the originality of the article (EU) resides, and thus it is important to dwell on them at length.

Phrased in ordinary language, we may refer to as 'obvious' something that is immediately fathomed, requiring no articulation to be sensed. It is so to speak, primitive, in the sense of not requiring, nor even being amendable, to a reduction to even more fundamental parts. What is obvious could vary from a specific fact, to some general kind of reasoning.

[1] At some point Poincaré starts to be 'silly', as he seems to indicate that the subconcious may be able to check 'all' the possible combinations.

What is 'evident' is something other. It is not immediate, but follows from a chain of reasoning, a kind of calculation if you like. The sense of the 'obvious' is like the sense of the 'color red', it is a qualia, a component of our individual consciousness, and hence not communicable. 'Evidence' on the other hand primarily involves not so much calculation, as communication. It is a shared reality, in which we calibrate our experiences with some other, and agree on an 'objective truth'. To confuse matters slightly, the 'other', could be ourself, and thus the 'evident' involves a kind of interior monologue.

To illustrate those two concepts we may think of two simple arithmetical calculations. That '$1+1=2$' we may find obvious. It is felt to be true, to be a fact, just like that rose in front of you has the color red. It is not something you have to convince yourself of. It is irreducible, it is an integral component of 'what is' and you find yourself stymied if you try to think of it being otherwise. On the other hand that ' $1325+2743=4068$' is not as 'obvious', to establish the truth you need to do a 'calculation'. The calculation is simple and straightforward, but it is a calculation, the answer does not 'pop up', and it does not seem to be as integral part of 'what is', like the obvious $1=1=2$. However, if you would like to explain to somebody that $1+1=2$ you find yourself in a quandaray. It is 'obvious' but that very 'obviousness' is not communicable. On the other hand it is much easier to explain to somebody that $1325+2743=4068$, the 'evident' is communicable.

The author is inspired by Jurgen Habermas, and provides two quotations from him. Both quotations are incomprehensible to me, which in part may be a consequence of they being extracted from a wider context, which would have provided clues and prepared the reader; and in part may be due to a regrettable, if understandable, desire to obfuscate, which may be mistaken for profundity. The author does, however, amplify and explain the two quotes, as to enable me to formulate my own interpretation of the nature and distinction of the two concepts. I may very well have missed some subtle point, as I am unable to understand the initial definition of the author, which is quoted verbatim below.

> När något uppfattas som säkert - som uppenbart eller evident - gäller detta i ett visst sammanhang och detta förhållande behöver inte gälla i ett annat sammanhang. Om betingelserna för förhållandet ej fokuseras utgör dessa en förgivettagen och konstant grund i vilken vissheten framstår som uppebar. Om däremot betingelserna fokuseras öppnas möjligheten för variation och vissheten grundas på en diskursiv evidens. Det som framstår som säkert i det förra fallet kallar vi det uppenbara och i det senare fallet det evidenta[2]

Until further instructed, I will employ the definitions, and distinctions, I have phrased naively above. They are, however, sufficient to continue the discussion, and also to throw some specific lights on the musings of Poincaré.

Poincaré refers to those sudden flashes of insights as certitudes. He knows that they are true, in the sense of a gut-feeling. But they require articulation (to be written down) and verification (to check by calculation, and/or routine logical reasoning to fit with the bigger picture.) Should one say that what is revealed is the

[2]When something is experienced as certain - be it obvious or evident - this is valid in a certain context, and this something may not hold in another context. If the conditions of this something are not focused, they constitute a given and constant setting in which certitude appears as obvious. On the other hand if the conditions are focused, there emerges the possibility of variation and the certitude is founded on a discursive evidence. What appears as certain in the first case, we call the 'obvious', and in the second case the 'evident'

'obvious' and what follows later is the looking for 'evidence' to persuade yourself? That truth is not inevitable, nor primitive in any sense; and until that very moment, we did well contemplate the world ('what is') without being aware of its truth, while admittedly it might be hard to do so afterwards. And why do we need the verification? Is it to convince ourselves, or to prepare for the communication to others? Experience shows that for this highly complex kind of 'obviousness' we can be decieved. Verification fails to verify, and does point out fallacies. But if so, why do we abandon our direct sense of the feeling of truth, for a process, which is more circumspect, and never leads to the same kind of understanding? This seems to be part of a principal choice. When there is a conflict, we almost invariably embrace the evident truth. Could that be a submission to objective reality, a kind of faith in the supreme reliability of the method?

The serious Philosophers of any era, at least until the mid-20th century, have been intrigued by mathematics (much to the frustration of latter day students). The source for this interest has been epistemological. Mathematics has been seen as the most solid source of secure knowledge, free from the vagaries of our defective senses, or the arbritariness of mere opinion. In mathematics we take nothing on faith, save what is inevitably (and hence inexplicably) obvious, and argue by principles of thought beyond reproach. (In fact our imagination does not suffice to imagine worlds where this does not hold true, unlike the variations of our physcial word, as conveyed by our senses, in which we delight to vary what seems merely accidental). Thus mathematics constitute the ultimate in objective knowledge, independant of time and other external circumstances. Implicit in this assesment is that mathematics is important beyond itself, no one worries whether the rules of chess are true, or whether the moves are made legally. Mathematics is important, as it somehow describes the essence of the world, and is not a formal game.

In a strict sense this has not been entirely uncontroversial. The Platonic attitude that mathematics somehow exists, in fact that this, the most cerebral of all activities, is really what constitute the eternal world of ideas, whose shadows constitute the world of our senses; is an attractive one, especially to mathematicians; and in fact, whether articulated or not, this is the philosophy of most working mathematicians. This attitude may seem naive to sophisticated non-mathematicians (as well as some mathematicians), and the meta-view of mathematics, as games played with arbitrary set of axioms, subjected only to their internal consistencies, has been known as the formal school. From the point of view of the formalists, mathematics is just a collection of tautologies, stringed along by syllogisms, and hence ultimately a branch of logic. This is not the place to reiterate the well-known history of how mathematical formalism foundered during the first third of the century, and how this crisis was resolved. However, the view of mathematics, as based on the axiomatic method, is a fundamental one, and well-known to concerned non-mathematicians. It is the basis of the shared trust in the 'infallability' of the mathematical method, and although ultimately it has to be taken on faith (like religious dogma) it nevertheless has been amply borne out by experience, as exemplified by the uncontroversial nature of mathematical facts, as well as by its oft quoted unreasonable effectiveness in its description of our so called physical reality. But any serious inquiry into the nature of mathematical activity, must go beyond the mere syntax. Proofs in mathematics are seldom if ever, broken down into small, logically impeccable steps; and from a pragmatic mathematical point of view, their nature is less that of mere verification as exemplified by calculation, than as means of persuasion and instruction[3].

To rely solely on introspection, or on received philosophical wisdom, is not enough. In order to bring the discussion down to something specific, and especially to matters distinctively mathematical, there is a need to discuss concrete cases. All human persuasions ultimately rest upon logical reasoning, stringing arguments along, starting from known premisses; and all cereberal human activity consists in trying out various combinations, subjecting them to different kinds of tests. In order to make a discussion of mathematical thinking fruitful, this general aspect will always have to be kept in the back of the mind, as well as the need to focus on the way mathematics differ from other kinds of thinking.

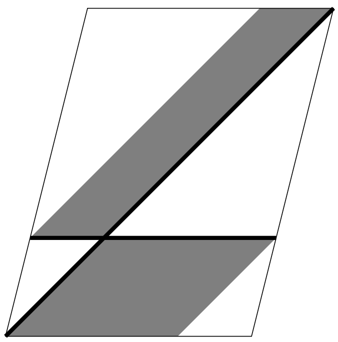
Videos

How do we get at the thinking processes? From a strictly epistemological point of view, thinking may quite well be inaccessible; however it seems reasonable to take a pragmatic point of view, as we do in ordinary social life, to infer at least some fragments of the underlying thinking from the utterances we are directly privy to. Those may be oral, or written, as in examinations, and we often assume that we can get quite a lot out of it, at least whether somebody 'thought right' and 'understood', or merely 'thought wrong' or most likely 'not at all'. It seems generally believed that the spoken word is more immediate than the written, and that its 'spontaneity' would give clues to what is going on 'inside'. Thus it has become 'popular' in studies of students learning and general acquisition to make videos of their attempts. In order to make the recordings verbal, and not just opaque pantomines, the students are made to interact, either among themselves, so called 'group work' or with an 'instructor'. Of those two methods, one may think that the former is superior, after all there are no external promptings from somebody 'who knows the answers' but a supposedly honest effort to share your thinking with others. On the other hand group discussions may drift off, and a 'gentle' instructor may be needed to steer it into more fruitful waters. One thing is clear. The studies are made at a very high level, and thus the naivity of the controls may be forgiven. A video-recording documents at a very high information level, the great surplus of which is simply thrown away. The main purpose is to get new ideas, not to document thinking per se, and in this light, simple notes by the instructor or observer should suffice. However, videos is a convenience, but one should beware of adducing to the practice a belief in objective science. A video recording of a student struggling to come to grips with some mathematics, is not a high-speed recording, able to reveal to the eye, (like the old controversy whether galloping horses always had at least one hoof on the ground) what is ordinarily hidden to it. One should also note that the transcription of a video-recording is a very time-consuming project, and that the straightforward reading of it can be quite painful. I have personally a low tolerance for watching people beating around the bush, not grasping what is crying out to be seized. This is, however, a personal shortcoming, and I do not doubt that there are people far more patient, who can get out of a transcript quite a lot of hidden information, like an experienced criminal interrogator. If the introduction

[3]The question of rigor is however a fundamental one, and if the hand-waving becomes too liberal, trouble lurks ahead. There is an interesting history to this, taking up the 19th century and the beginning of the next, but in this essay there is no space for it.

of video documentations is going to be a major method of mathematical didactics research, some sophistication of the practice is needed to be developed. At this stage I believe this is not so pressing, as there is a lot of work to be done, before proper questions and studies can be devised. In EU the main scientific content, as opposed to the philosophical, consists of a video-study[4], to be discussed at length below. The object of that study is to illustrate in a 'real-life situation' the different concepts of 'the obvious' and 'the evident' which is the theme of the article.

The Study



Five prospective math-teachers are presented by a simple geometrical problem, as illustrated by the figure on the left. A parallelogram is given, a diagonal drawn, and an arbitrary line parallel to the base. (Both of them marked by bold lines). By drawing lines parallel to the diagonal, two shaded figures arise, and the students are asked about the relationships between the areas. The students have been taught concepts like familiarity and congruences of triangles, but have never encountered a similar problem. What is unusual is the arbitrary way the horizontal line has been drawn.

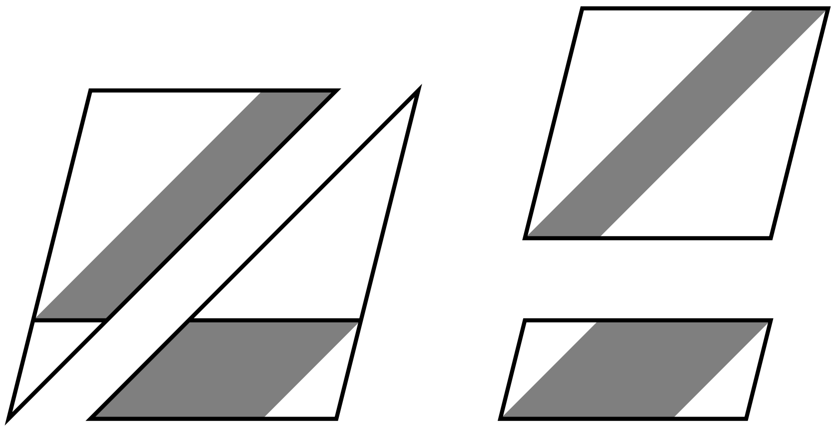
A professional mathematician (like myself) solves it in about ten to fifteen seconds by noting that the respective heights are inversely proportional to the bases. All the details cannot be explicated in the short flash, but the conviction that the details are nothing but just details, no doubt based on experience, forms quickly.

The female student T. professes confusion. She is disturbed by the arbitrary position of the horizontal bar. 'I try everything, randomly, desperately *(hej vilt)*' she seems to say, but of course she has a strategy, looking for triangles, with the ultimate purpose to combine triangles with the shaded figures, as to make something obviously equal in area. This does presupposes that she has, at least subconsciously, formed the hypothesis that the two areas are equal, an assumption that is far from obvious. What is obvious though from the figure, are the similarity, and in certain cases, the congruences of triangles. Now it is one thing to feel and believe that two triangles are similar (or congruent), quite another thing to 'prove' that they are. The author makes a distinction between 'practical evidence' and 'theoretical'. The problem-solver is (more or less desperately) looking for clues, and visual clues are simple to find. She is also capable of invoking results from geometry, like the criteria for congruence, or similarity of triangles, which the author refers to as theoretical evidence. An interesting question is what relation those two types of evidence has for the girl. The author does not discuss it. To me one thing seems very plausible, namely that it is the practical evidence which is the guiding principle, and she uses

[4]In fact, I was later informed by the author, a simple tape-recording was used instead. This does however not affect the thrust of the previous arguments, as most, if not all, video-studies in didactical contexts, seem only to take into account what people say, or try to say, not on their body-language

her theoretical evidens only to verify. And why does she felel the need to verify? Is it because she is unsure of herself? That the eye can be fooled, that figures are far from perfect[5]. or is it because she thinks of this as a mathematical game, or more or less equivalently, that she needs to justify her claims, not so much to herself, as to the instructor. Such deeper questions cannot be resolved by a simple video session, nor I fear, by straightforward introspection. The instructor (and author) shows at least some impatience as indicated by his questions as well as his lengthy analysis of why it takes her so long to jump to the right conclusion.[6].

Her final conclusion hinges upon the possibility to view the figure in two different ways, as illustrated below.



The partition of the figure to the left is obviously the one that is needed, while the figure on the right is a dead-end for her purposes. the most intriguing question is not why it took her so long to come up with the right partition, as she was trying out all kinds of combinations wildly, when there are so few ways of dividing the figure; but why she did have something like this is mind from the start. This question will hardly be answered by video documentation, only the second affords some possibility of illumination. So what can we glean from the authors analysis of her conversation? One plausible interpretation is that the partitions of the figures are not done on any systematic basis, maybe not even articulated, but that she is groping for clues (fixing her attention either on the horizontal bar (bad) or the diagonal (good)) waiting for the thing to 'pop up'.

The author describes her solution as synthetic, or more precisely that she used synthetic evidens to get to her conclusion. This solution has many advantages. It is simple, and it is compelling. Even a small child might possibly understand it, as it would certainly perceive the corresponding triangles, but from that to conclude that the residual shaded areas need to be the same, may be a conclusion to sophisticated

[5]Even computer-drawn figures like the one presented her. I suspect that the original figure was drawn by hand

[6]The fact that it takes so long, can be seen as a bonus, as it allows a focusing on the process, and a hope that its slow motion will allow the observer to pin-point crucial steps, which ordinarily are subsumed

for it.[7] What is slightly bothersome is that she needs to express the relationships algebraically, to get to the obvious conclusion. Whether this was just a concession to playing a mathematical game, or instrumental in her realisation, we can only speculate upon. My guess is that in the end, she would realise, that it is a step she could dispense with.

As noted the synthetic method has advantages. It makes the whole problem into a kind of puzzle[8], and mathematical sophistication, even rudimentary, unnecessary. However, it also has some disadvantages. It is ad-hoc. You might see it immediately, or you might take quite a long time to see it (or maybe not at all, unless it is pointed out to you). The solution reminds me of peculiar pictures, who at first do not make sense, and then suddenly, they 'pop up' in front of you. This is an altogether different kind of line of research, which ought to have been studied by visual psychologists.[9]

The male student M. has another approach. From the interview one gets the impression that the student thinks in terms of problems as standard obstacles to be dispensed with by a collection of recipes. He is taken aback by the unfamiliarity of the problem, and needs to have the construction repeated to him slowly, as to get a grip on the situation. He forms a hypothesis that the two areas are the same, but is unable, (or unwilling) to produce any motivation for his guess. Maybe he thinks of this as a 'problem' in a school-context, and those are all more or less contrived, making equal areas a typical kind of answer you expect. The latter hypothesis seems to be borne out by his professed strategy of testing. If he solves an equation, he admits in an aside to the instructor, he tries out different values, based on his expectations of what the problem constructor may have had in mind.

Does M. understand the problem? I mean not just as a formal game. Does he have some intuition? What is area to him? He seems to lack the tools to handle this mathematically. How does one compare areas. He mentions that a ruler may be of help, but that is not elabourated upon. He finally resorts to cutting and redistribution. This seems to indicate that he has some sense of areas. Probably a sense of area predating actual mathematical instruction. He does however introduce a new idea, namely that of moving the horizontal bar. The relationship should be independant of its position, hence nothing should change when the line is moving. But due to his rudimentary mathematical sophistication he cannot pursue this idea. The instructor is gently trying to lead him to the solution along synthetic lines but with scarce success. His attempts fizzle out. The theoretical evidence that his mathematical education should have provided him does not occur to him. Maybe

[7] This could be a subject of a study by itself, and has no doubt been a study in the past. Maybe the problem may even be discussed in the works of Piaget.

[8] I recall a puzzle presented by Gösta Knutsson (familiar to Swedes above a certain age) on Swedish (public) TV in the early 60's. A square was drawn, an inscribed circle, and then an inscribed square, with sides parallel to the first one. The relationship between the areas was asked for, and the viewers were encourgaed to send in solutions, on the one condition that no mathematics was used. My father, a math-teacher, wondered with some exasperation, how that would be possible. Later on a solution was devised, essentially involving the rotation of the innermost square by a right turn, and presented by a somewhat elaborate wooden model, with foldable parts.

[9] My mother-in-law has had on her fridge door for twenty years a photograph of some snow-covered spruces. The picture is black and white, with a high contrast. The casual observer (like myself) may more or less instantaneously behold the face of Jesus, while my poor mother-in-law has yet to make it out.

his mathematical knowledge is narrowly tied to a very specific context, once he encounters an atypical problem, let alone a real-life situation, he is at loss.[10] Now the purpose of those encounters are not to judge people, but to discover how they think, thus it is important that a great variety of students are considered. This does not mean that the studies are not depressing. The conclusion we can draw from this case is that there seem to be mathematical concepts that people acquire without the need for a mathematical education. 'Area' seems to be one such.

The female student C. wants to find a solution based on, in the parlance of the author - analytic evidence. She is at first overwhelmed by the problem. There is a multitude of lines intersecting at all kids of places. She does not know where to begin, and she gets into a panic, finding that she cannot think at all. The instrcutor gently tries to steer her into a synthetic solution, by pointing out the diagonal, and how it subdivides the figure into two equal parts. She admits it, but it does not stimulate her thought. She starts to think of areas, as something you compute, remembering the formula. But her attempts to get the right relations between the entities, leads into a messy picture, and a hoist of confusing notations. She is just unable to keep everything straight.

The interesting thing here is whether she has a formal view of area, as something that pops up as a result of a computation, derived from a formula, she no doubt has learned from heart, or whether she can relate it to congruent triangles? The instructor, somewhat exasperated one surmises, tries to point out this syntehtic conception of area. Could it be that she has no intuitive concept of area, or if she has, that the context of a mathematical problem, makes that intuition irrelevant?

The female student A. likewise wants an analytic solution, in spite of the instructors insistent attempts to steer her into a synthetic. Judging from the transcript her efforts are not as frantic as those of C. but this is apparently not due to a better command, only to a lack of energy, and predictably her efforts fizzle out. The author/instructor laments in his analysis that it is not clear whether she even understands or accepts the synthetic solution, once she has it pointed out to her. This is a very hard claim to test, but nevertheless indicative of the lengths to which we are willing to carry our interpretations of the inner lives of our fellow humans.

Finally the male student J. seems to have a clear strategy, seeking, in the words of the author, a synthetic solution. He rearranges triangles, tries many combinations, but nevertheless he gets stuck, and needs some gentle prompting from the instructor, to find the solution. His mind is prepared, so to speak, and thus he has no difficulties recognising it, when it presents itself. The transcript is rather short, compared to the others, and one cannot entirely avoid the suspicion that the author is getting fatigued.

Conclusions

What does this study, which obviously has taken quite a lot of time and effort to execute, really tell us? First, what did the author expect? Why did he choose that particular problem for his subjects to work on? Was it chosen more or less

[10]For a mathematician, he has in no way internalized his knowledge, he is unable to apply it. It remains dead and formal, with no empowering benefits of his education, which, at least judging form this example alone, was completly wasted on him. One naively wonders, how he will be able to educate others, when he was unable to benefit from education himself.

randomly, or was there some thought behind? The author does not provide any explicit clues, so I have to do some second-guessing.

Euclidean Geometry was a time-honored subject in mathematical instructions all over the Western World. The ulterior object was to introduce to the student 'deductive' thinking, not geometric knowledge per se. The idea being to show the power of pure thinking, and its superiority over mere guessing. In particular the structure of a proof, many examples of which were shown to the students, was held up as a model of disciplined reasoning. In addition to the illustrative purposes, it also gave the students tools of discovery, and thus it provided the context for a great variety of elementary problems, taxing their ingenuity.

To some students, (I fear a minority), the encounter was a philosophical revelation[11], to others it gave the tools for problem-solving. Of course both aspects were not contradictory, in fact only an appreciation of both, allowed a deeper understanding. Anyway, through practice, a fair number of students became quite proficient, and I would guess that they would have had no problems with the task, all of them providing an analytic solution. Modern students have no such prior practice, although I guess they have some familiarity with the concepts of uniformity and congruences, but that may have been exposed to these fairly late in their curriculums. The point of the problem is that it can be solved as an ordinary puzzle, using intuitive concepts like areas. It is fairly natural to try and compare two areas by dividing and rearranging pieces, the leap which is required here, is to realise that one can also compare areas, by adducing to them other areas, to build up figures that can be compared. One of the great drawbacks of many studies like this is that one does not really know the background of the students. Really how much geometry have they been exposed to, and in particular have they ever encountered this technique before? If not, we are witnessing some processes of genuine discovery.

The intervention of the instructor is very problematic. The instructor is there both to elicit comments, and to gently guide, and thus inadvertedly to intefere. He tries to stay neutral, but this is impossible. More seriously though, is the improvised character of his verbal intercourse. Thus one can dismiss the study as being a controlled experiment, in fact interesting controlled experiments may never have been done in mathematical didactics, at least not on a higher level. Didacticians should have much to learn from the psychologists, but the time for such experiments may not yet have come. Thus the purpose of the study is illustrative, with a hope of describing thought processes, and reveal phenomena, not yet suspected, to serve as inspirations for further studies.

But how do you get at thought processes, given the fact that those are even opaque to the subjects themselves? The video recordings do give, as remarked before, an incredible amount of information (in the technical sense of bytes) most of which are discarded in the tedious and time-consuming process of transcription. The first role of the transcription is to make the data printable. Transcription inevitably involves editing, and the second goal is to present readable narratives,

[11]The axiomatic method, as a form of seeking 'truth', became clear to me around the age of fourteen. I remember formulating my 'philosophy' on the subject, during a vacation to the Adriatic coast. Religion, I reasoned, mainly focued on the axioms, while the technicians were mostly interested in the results; however, both of those aspects of truth were irrelevant, the real fascination resided in the bridge given by thinking, and therein was the essence of mathematics, surely the most exalted activity imaginable. In retrospect one may here discern the embryo of formalism, a phase through which many philosophically minded mathematicians may have to pass.

with enough detail, as to enable the interested, and sceptical reader, to check the comments of the author. Thus in addition to the orderly sequence of well-articulated statements, there are lots of gruntings (*Oj,oj,oj,ja,ja,ja,mmmmmmm*) as well as the indication of silences and/or omissions (....). Those are never explicitly used in the analysis, but a discerning reader may possibly put them to unprecedented use.

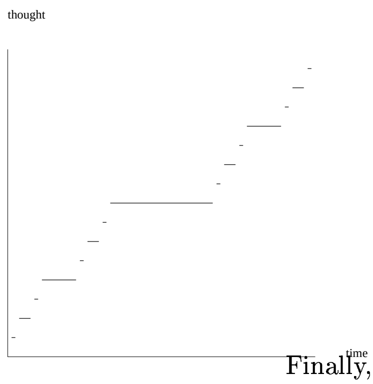
There is a distinction between verifying what is felt as true, and to discover something new. Many people (including most mathematicians at a lecture) think of proofs as a verification after the fact. (To this the practice of first stating a theorem, then proving it, certainly contributes) In that sense proofs, if necessary, becomes somewhat tedious. In elementary instruction, theorems may even be intuitively obvious, and thus the demand for proof, may be felt as pedantic hairsplitting. But a proof is nothing but a documented reasoning, and many proofs antedate their theorems. Mathematical reasoning is a tool, not just for justification but for discovery.

Is it clear to the students that the two shaded parallelograms have equal area? They look quite different. It is one thing to verify what seems obvious. If that is the case, the problem that faces the student is not to convince himself, but to convince the instructor that he has learned the game of formal verification, with a kind of sinking feeling, that if he has not, he fails the rigorous standards of mathematics, and what that to him seems clear, may just be an ephemeral mirage. This may contribute to mathematical alienation. If it is not clear to the students that the areas are equal, do they care? This may seem to be a frivolous question, but I find it crucial. If they do not really care, they may be confronted with a test of ability; but it is not something that arouses their curiosity. Without genuine curiosity, there is little constructive thought. Combinations do not occur, the mind is a blank, and the documentation of 'empty' thought processes is necessarily sterile. Why should they be curious? The question has no obvious practical applications, although it might be set in a suggestive context (a challenge for the tester). If they already has some mastery of elementary geometric techniques, their curiosity might be fuelled simply by the opportunity to wield those tools. (This is what drives many professional mathematicians). When there is curiosity, there are ideas, combinations, associations.

In the thought fragments which are presented in the study it is clear that thought is not linear. It is not like a calculation, that inexorably, if often in a circumspect way, drives towards its conclusion. In thought there are many dead-ends, avenues pursued, time wasted. Sometimes you are very near to the resolution, but you fail to take the crucial step, and drift away. When all is said and done, you realise that most of the thought was irrelevant, the traces of which can safely be disregarded, and of course thus never appear in a final write-up. In this mathematics does not differ from other thought processes involved in all kinds of problem solving. But is there a way of avoiding those dead-ends? Is there a way of systematically thinking of problems, without wasting your time on red herrings, and wild geese? It is a meta-philosophical contention that this is not possible, at least not in mathematics. There is no super-algorithm that allows us to tackle problems systematically with the efficiency of a calculation. Phrased differently: There is no way we will ever be able to understand the thinking of mathematics in a way, that will enable us to do mathematics more efficiently. There are no such shortcuts. The understanding of mathematical thought will be of reflexive and descriptive nature, never proscriptive in any specific sense of the word. It is true that efforts have been made, notably by Polya, to not only describe the processes of mathematical problem-solving, but also to instruct. I claim that such efforts are ultimately futile, there is no short-cut, that avoids the practice of actually solving problems. Such general and far reaching claims are obviously not subject to formal proofs, they are in the nature of faith, religious if you so prefer.

To step down from the heights (and follies?) of lofty philosophical proclamations, one can also take a more pragmatic view of the tangled thought processes involved in trying to find a solution. Those dead-ends may be dead-ends as far as the actual problem is concerned, but they do not disappear without trace, but more often teaches you something than not. Even if an attempt is useless for the moment, it might be valuable in the future: and even if not, negative experiences are not always worthless. It might be of some use, to recognise sterile ways, and to be able to avoid them in the future. Repeated exposure to problems, especially if frought with frustration, could be very educational. In fact mathematics knows of no better way of learning and appreciating mathematics. Unlike the arts, just watching is not enough, you have to do it, to enjoy it.

We all agree that thought processes are not linear, but tangled. Still they take place in linear time. Some thoughts come before others, although in retrospect it might be very difficult to remember the exact order, as it is not intrinsic to the process, but is an imposition. In the same way a proof is also linear. The arguments follow in a specific order. This is also a convention, imposed by time, and hence very difficult to escape. But 'understanding' is not linear. I have previously likened the flash of insight, to the making of visual sense. A picture we take into one glance[12] All the pieces fit together in a kind of timeless manner.



Devil's staircase

Finally, I am fond of pointing out the disconnected du-

[12] Obviously this is not true, cannot be true. Visual perecption is subject to the same temporal constraints as everything else. This has been demonstrated over and over again. Recent visitors to the National Gallery at London could enjoy an exhibit ('Telling Time'), in which the eye movements of volunteers were recorded, and displayed on the actual picture that was being viewed. A seemingly random walk, interspersed with brief moments of rest (of various durations, denoted by circles, with proportional radi) covered some part of the canvas. Not only must the mind process information temporally, i.e. following an ordering, the eye itself can only focus on a very minute part, the rest of the field is blurry. (Clearly the latter is a consequence of the former). Thus we 'make up' most of the picture, filling in details, a task for which we are abetted by long experience. However this is not the way we experience the picture, all of this is done quickly and automatically, and subconsciously. In fact research in clinical psychology seems to indicate that the eye usurps by far the major part of the processing of sensory input, which may be why most of us fear blindness more than any other deprivation of the senses, as this most closely resembles our actual idea of death (total sensory deprivation).

ration of thought. We think of something, but not all the time. In fact within each time-interval, there are large segments, in which we are distracted. The mathematical metaphore that comes to mind is that of the Cantor set. With tongue in cheek, we may claim that the actual duration of 'real' thinking has zero measure[13] but nevertheless we make progress. (One thinks of the Devils staircase, a function flat almost everywhere (i.e. when we do not think) yet managing to increase.) The point of the metaphore is to point out that thinking involves spikes, or leaps, if you prefer. Those can be of various magnitudes, but they all involve some measure of 'discontinuity'. A flash of insight lifts, and there is a kind of break between before and after. The solution of a problem typically involves a few (lucky) breaks (to make a pun), everything else is just routine. What distinguishes the experienced mathematician from the neophyte is that he needs fewer breaks, as his capacity for interpolation is much more developed. Still nothing that I have described here distinguished mathematics qualitively from any other kind of thinking, except that the phenomenon might be most pronounced in mathematics.

So what is the point of such studies, we have now considered? It would be naive, or at least premature, to look for quantative results. In that sense a single good example may as well serve our cause, as a dozen indifferent ones. The ultimate purpose of such studies is to investigate to what extent one may make thought processes transparent. Many of the aspects of thought I have brought up in the discussion above, are not considered in the paper (EU). One should, however, never fault an author for what he does not (as restriction of inquiry is the first step that has to be taken in any serious investigation), but for what he does. So what is the purpose of this study? The illustration of the contrasting concepts of the obvious and the evident, and, as it turns out, with special emphasis on the various kinds of evidence that are invoked.

I must admit that I find very little in the study which illustrates the difference between 'the obvious' and 'the evident'. The author notes that all the candidates start from something they 'know' and can 'trust'. This is hardly an original observation, but is almost tautological. Whatever you do, you must start somewhere to get your bearings. If this refers to 'the obvious' it is a poor illustration. It had been otherwise if the candidates had in general started from the premise that the areas had been equal.[14] Thus most of the discussion concerns various forms of evidence, which the author divides into practical and theoretical, and the latter is subjected to a further subdivision into synthetic and analytic. It is clearly here we should look for the meat of the article, wedding a general philosophical notion, with a very specific empirical input.

To start out with so called 'practical evidence'. This is tied with the notion of mathematics and the real world. The figure exists on the paper, you could in principle cut it out. Thus mathematical concepts like area, are infused with

[13]From a strictly mathematical point of view, this is nonsense. Every mathematician knows that you can make Cantor-like constructions , whose complement has arbitrarily small measure. But the example is not intended to be literally mathematical, just metaphorical.

[14]In fact, and this is a weakness, it is not clear from the artcle, whether the students believe that the areas are equal (I get the impression they do not) and hence that their task is to 'prove it', or whether they have no idea, and they have to find out. There is a big difference of studying students idea of required 'rigor', and their ability to discover. In the first case one is concerned with retroactive reasoning, in the latter with explorative. When we speak of students 'not understanding' proofs we think of the former.

meaning. The notion of 'obvious' may thus be thought of as 'facts' and 'relations' practically corroborated.[15]. Theoretical evidens, on the other hand, comes with education. Education is supposed to give you tools. The tools mathematics, or more specifically euclidean geometry, with its results and principles of deductive thought, constitute both a tool to seek new results, as well as a paradigm in which to communicate them, as well as verify them.[16] The author divides theoretical evidence into analytic and synthetic. Those are given very specific interpretations in the study, namely the analytic refers to finding relations between bases and heights, in order to exploit a standard formula; while the synthetic concern the areas directly, and how they add up with additional triangles to form entities directly comparable. The author notes that in the first case the stdenst concentrate on the lines, in the second on what is enclosed by the lines. The meaning of a diagonal is very different in the two approaches.

Now the problem is supposed to be illustrative of a more geenral situation, and then it would be interesting to learn from the author the deeper distinction between 'analytic' and 'synthetic' and whether this dualism is a fruitful conecpt.

My interpretation is that 'analytic' refers to the use of a 'machine'. One proceeds without real understanding. A student could have been told of the notion of say 'whizz'[17] of a figure, and how that is computed in specific cases. The analytic method presupposes familiarity with techniques, that have to have been mastered. While 'synthethic' would refer to a more ad hoc and intuitive approach, involving an intuitive understanding of the concepts, and the associations that engenders

Now it is not possible to make a real clear distinction between the two. In many approaches, a mathematician have little choice but to follow the 'analytic' method, on the other hand each real breakthrough is affected by stepping back, reflecting on the problem, setting it into a wider context, and maybe getting a glimpse of the intuitive, hidden, meaning of a formal concept. I am not sure whether the author has this general distinction in mind, anyway it is not really clearly illustrated in the study.

In what sense is Mathematics special?

If you look up mathematics in a dictionary, you will find something to the effect of 'an abstract science concerend with number, quantity and space'. It is hard to fault this definition, especially if you are confined to a line, on the other hand it gives almost no inkling of the great variety of mathematics, and what diversity resides behind those non-descript labels.

In the typical definition above there are in addition to the objects of the study, (numbers, quantities and space) the qualifier - abstract. What is the essence of mathematics? The objects, or the (abstract) way they are studied? Abstractness is surely one of the essences of mathematics, and it is tempting to conclude, that mathematics is foremost a way of reasoning, regardless of the objects of the reasoning. This may explain the effectiveness of mathematical thinking, not just in

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[15] I am not really happy with this infusion the author makes with the notion of 'obvious' into the notion of the 'evident' Were those not supposed to be kept separate?

[16] One may think that mathematics is a pure activity, but in practice, something is not proved, unless others can be persuaded.

[17] A non-sense word to preclude all a priori understanding of the concept, unlike choices like 'area', 'spread', 'content', 'extension' etc

the proper realm of mathematics, and also the perceived usefulness of mathematical study at school intended to sharpen the powers of reasoning in general. This view is quite attractive, at least to a mathematician, and much of what we have been discussing in general terms seem to bear this out. Mathematics is a way of finding fruitful lines of arguments, to discern the correct from the false, to combine different strains of thought. Is all rational thinking (worth its name) mathematical in nature? I would say no, and I would claim that it is important to distinguish mathematics from other kinds of reasoning, with which it shares many superficial traits. In fact I believe by comparing mathematical thinking with other kinds of thinking, our imagination is jogged, and it would help us to focus on what is really mathematical, thereby making our studies more specific and interesting. Just thinking in general terms does not help, it is only by being confronted by contrasts, you realise what stands out.

First consider subject matter. What is the difference between mathematics and physics. To the layman, exposed to the mathematical language of a physicist, there may be no difference whatsoever. I think that there is a definite difference, but it is more subtle than you would normally think.

To turn things around, why is not Euclidean geometry considered physics? It studies space, but space is physical. One may argue that Euclid is concerned with an ideal space, a notion residing in a Platonic heaven: and that one should distinguish between pure geometry and applied. One may argue with Kant, that space is not primarily physical, it is an inherent feature of our mind, a purely cerebral structure we use to organise physical sensation. The axioms of Euclid are mainly those of pure logical thinking, to which a few geometric postulates have been added[18]. The most famous of them is the parallel postulate, which differ from the other, as being subject to doubt. The source of this doubt is its non-local structure, as it explicitly calls for an indefinite stretching of lines. With the advent of the non-Euclidean option almost two hundred years ago, for the first time, geometry as a model of space, emerged as a concept. Before that it is hard to argue that there really was a distinction between mathematical and physical space. (And of course it is fruitless to try to ascertain what Euclid 'really' thought).

But if there was no difference between mathematical space and physical space in the past, how come not more of physics was assumed into mathematics? Mechanics is a good example. You can easily think of it as pure mathematics, starting with Euclidean space, but adding to it a few more concepts and postulates. In fact from the modern point of view, classical mechanics can be treated as a purely mathematical subject.[19] The point is that mechanics is much more complicated than mere geometry (even if it formally can be subsumed in the latter). The postulates of elementary geometry are 'obvious', not so in mechanics. That the free falling body accelerates at a constant rate is not something you can divine, this has to infered by experiment. It is quite another thing, as Newton did, to devise some simple postulates (the three laws of Newton) looking like those you find in Euclid,

[18]In the view of Kant, fundamental features of space, are as inalienable from the human psyche, as rules of syllogisms

[19]This is of course not true for most part of physics, which is a Natural Science, proceeding by hypotheses, measuring and inductive thinking, in which mathematics play a crucial descriptive role, but only secondary in its capacity as constituting various models. One does not divine things in physics from pure thinking, except in modern string theory, where empirical testing is not an option, and the most reliable guide is mathematical beauty!

from which those can be derived. It is quite remarkable, that Newton fashioned his physical theory on Euclid, as if he thought of it as an extension of the latter.[20] More interestingly still, although there is a correlation between mathematical prowess and physical (which should surprise few) it is quite common that someone with an impeccable geometric intuition, lacks a physical. What is the difference? Is it just a matter of physics being more complicated? Or a matter of taste (as mechanics is not part of mathematics, someone interested in mathematics does not need to stake claims there)? It would be interesting to compare a study, like the above, but now concerned with a mechanical problem. How would the thought processes differ? In the geometrical situation we observed a dearth of ideas, would the field of associations be richer (but of course not necessarily more correct) in the mechanical case? Such a study might help to highlight what is more specifically mathematical in the geometric case.

Some people are mechanically gifted. They can solve practical mechanical problems in elegant ways. Often such people are dismissed by more verbally attuned people as 'good with their hands'. But it is not primarily a question of being manually more agile, in fact manual agility is rather an effect not a cause. What distinguishes those people is their clear thinking of the issues. They also form long chains of reasoning, they argue by principles; but their activity is not guided by words, but by tactile means.[21] Even a study of students trying to do a practical mechanical problem, like devising a machine, which is a tangibly physical manifestation, of pure thoughts, could be instructive, when compared to a mathematical task.

The attempts to manifest mathematics tangibly, translating proofs into machines so to speak, has a long venerable history, starting with various acabuses, via calculating machines, culminating in electronic computers. Those machines are in a sense nothing but mathematics made flesh. An externalization of the thinking process, may free those from the vagaries of the human mind, such as limited memory, fatigue, psychological blocks, inattention, irrelevant distractions. However, there is a huge difference between the externalization of the products of the human mind, and the mind itself, which is a lesson many want to impart to the proponents of Artificial Intelligence.

Until the 20th century, clocks were the most intricate mechanical gadgets around. It is instructive to look at a watch created by Harrison[22]. Inside such a watch, one

[20] It is often pointed out that Newton derived his results through his 'calculus', but when he wrote them up in 'Principia' he wrote them in the style of Euclid, making them obscure in the process. However, doubt has been cast upon this story, reputedly told by Newton to Leibniz, it now looks as if Newton really was thinking in geometric terms, which makes our case stronger. This is of course a subject of mathematical history of ideas, rather than a study of students thought processes.

[21] It is a respectable, as well as deeply suggestive theory, that human intelligence (of which we are so proud) arose not through introspection or social intercourse, but through our ability to manipulate the world by our hands. Few animals, except our fellow monkeys, and the Elephant with its trunk, possess anything similar. Why should Dolphins be so intelligent? Bouncing beachballs is an opportunity offered very late in their evolutionary history, and to just a minority to boot. As to social intercourse, there are many social mammals, for which the 'reading' of others mind is very important. One may speculate wildly, that some of them possess 'social intelligence' on par, if not superior to those we have. But this is a tangent of unsustained inquiry, which quickly might lead to foolish fancy.

[22] The famous clock-maker, whose time-pieces, designed to withstand the vagaries of a stormy sea-voyage, were unparalleled at his time, and eventually won him longdue recognition in connec-

may find abandoned vestiges of constructions. Harrison was in a sense a programmer, the complicated machinery of his specially designed watches, were complicated programs, in which each part had to perfectly perform its functions. The modern programmer has it simpler, he does not have to worry about the hardware working properly. This leads us to the next step -programming.

Programming is really like building a machine, in a sense the machine is 'virtual' but it is a machine nevertheless. To program means to interact with 'reality'. Unlike mathematics, you get a direct feedback. Just as with a physical machine, it works or not. This feedback mechanism is very seductive, it means that unlike mathematics, when your thoughts may fizzle out, and disperse into a void, you are always egged on; when you are testing your programs, you are testing your thoughts, and of course you are almost within an hair-breadth of solving things. (I mathematics you may get stuck and that is it, in programming you never get stuck, unhinged perhaps.) Many mathematicians are quite amused by programming, as it gives to your thinking a physical manifestation, and thus a kind of tangible substance to them.[23]

To make a similar study of the thought processes when people are trying to program something could also be quite instructive. In that case one would also have the option of documenting the whole process of interaction with the computer directly.

Then there is the connections with games. Are not chess players masters of analytic thinking, constantly devising long chains of combinations of moves? In what way is mathematics different from chess. Once again it could be amusing to devise a study, when students describe their thought processes say in a game of chess. But even without such a study, it is of course easy to describe the differences in thinking, based solely on philosophical introspection.

First, and maybe foremost, there are two kinds of games. Those involving an opponent, and those that may be characterized as 'solitaires'. To make a direct comparison, we should obviously restrict to the latter. In a sense mathematics can be performed in a solipsistic universe, Chess or Go, or other games, obviously cannot.[24]. Chess has no (known) applications, in that sense it is a pure activity (like music). A Grand Master probably employs more 'brain power' than an average mathematician can ever hope to muster. A good chessplayer does not mechanically scan all kinds of moves (unlike the program), but is guided by some general (call it aesthetic) principles. On the other hand there are only a few levels in chess. Chess unlike mathematics does not go beyond itself. If the rules of chess would be changed ever so slightly, it would spell disaster[25] for the grandmaster. Just change

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tion with the practical problem of determining longitude, the story of which is no doubt familar to most readers.

[23]This might be worth an essay, just by itself. An essay that could treat mathematical experimentation, and also the contensious issue of mathematical proofs. Computers are calculators, and hence the connections with formal logic, are inescapable. Many logicians have also found a niche in computerscience. Also programming is much simpler than mathematics, and may crudely me compared to the solving of crossword puzzles as opposed to the writing of literature.

[24]Strictly speaking, in a solipsistic universe (which does not mean that 'you' are everything, which is just a matter of semantics; but that there are no other 'conscia' than your own) you can program a computer to play the game against you. It is then a philosophical question, to what extent that program goes beyond you.

[25]Something that needs to be checked, but could easily be done by some rather straightforward reprogramming, and a volunteering grandmaster or equivalent

the moves of a piece, or extend the chessboard. Mathematicians are quite a lot more robust. In fact one of the charms of mathematics is to change the parameters, and its most favourite gambit (as pointed out by the G.H.Hardy) to put everything it has in jeopardy. (Proof by contradiction). Some simpler games, mostly of the solitary kind, admit a complete mathematical solution. But there is a difference between playing a game, and stepping aside, and revealing its structure once and for all. (In fact to 'play' all of its possible games in one flash). The closest we could get to play 'mathematical games' would be to devise strange axiom systems and ask the students to prove all kinds of things. How would this work out? What strategies would they use in a world where they have no bearings? The deeper point of mathematics is that it is infused with meaning, and those who fail to infuse meaning in their mathematical activities, are doomed to be lost.

Finally somewhat frivolously maybe, one may bring up the issue of 'jokes'. The similiarity between a joke and a mathematical proof has often been pointed out, as both involve punchlines. Striking mathematical reasoning consist in putting something familiar into an unfamilar context (or vice versa). Just as in mathematics, to make good jokes (and appreciate them) presupposes not only a good command of the language, but also intimate knowledge of the context (social, political whatever). It is often claimed, especially by the proverbial man in the street, that whatever computers will be able to do, they will never be able to make jokes. I wonder whether there ever have been attempts to study how people think up jokes. It would not be easy, but there is no reason to think that the deeper and really interesting study of mathematical thought would be any easier.

I have now suggested a variety of other studies. It may seem to be a frivolous exercise, and nothing to be taken seriously. But I am serious, I believe that it might possibly give clues to the special nature of mathematics, by highlighting what it is not.

Göteborg, February 8, 2001