Chapter 1

Partitions of $\mathbf{R}^3$ into curves

(with Mattias Jonsson)

A general technique for obtaining partitions of $\mathbf{R}^3$ into curves satisfying various properties is presented. The method can be used to partition $\mathbf{R}^3$ into unlinked circles of radius one, which answers a question posed by Wilker, or into arbitrary collections of real analytic curves. We also apply the method to study the set of bijections of the open unit disk.

Introduction

It is well known that the plane is not a disjoint union of Jordan curves. In this paper we will discuss various ways to partition $\mathbf{R}^3$ into disjoint curves.

Our main results are:

Theorem (see Theorem 1.2.3). $\mathbf{R}^3$ *can be partitioned into unlinked congruent circles.*

This answers a question in [7].

Theorem (see Theorem 1.3.1). $\mathbf{R}^3$ *can be partitioned into isometric copies of any family of cardinality* c *of real analytic curves.*

We also obtain the following:

Theorem (see Theorem 1.5.2). *Define a metric* d *on the set of all bijections of the open unit disk* U *onto itself by* $d(f,g) = sup_{x \in U} |f(x) - g(x)|$. *The metric space so obtained is path connected.*

The paper is divided into five sections. In Section 1 we introduce the general technique. We present and extend some known results. In Section 2, we consider partitions of $\mathbf{R}^3$ into unlinked circles.

In Section 3, we replace the circles by arbitrary real analytic curves. In Section 4, we briefly discuss the situation in higher dimensions, and finally in Section 5 we apply the method to "bijection spaces" of plane sets.

While working on this paper we found that other mathematicians had already been studying some of these questions. Some of their results are included for completeness.

1.1 The general technique

In this section we present a general technique for partitioning a space into small subsets. As mentioned, the methods have been used before, e.g. in [2], [3] and [5]. However, we will extend this method to deal with situations where we put further restrictions on the partition (e.g. one circle of each positive radius).

By a *partition* of a set X we mean a collection of disjoint subsets of X whose union is X.

A *circle* in $\mathbf{R}^3$ is a set isometric to $\{x_1^2 + x_2^2 = r^2, x_3 = 0\}$, where r is the radius of the circle. Two circles are *congruent* if they have the same radius. These notions extend naturally to n-spheres in $\mathbf{R}^m$.

A *Jordan curve* is a homeomorphic image of a circle.

We let $\mathbf{R}^+$ and $\mathbf{Q}^+$ denote the positive real and rational numbers, respectively.

If X is a set then $|X|$ will denote its cardinality, and we denote $|\mathbf{R}|$ by c.

If A and B are sets then $A \sqcup B$ denotes their disjoint union.

It is possible to construct explicit partitions of $\mathbf{R}^3$ into circles. The following example is due to Szulkin [6].

Theorem 1.1.1 *[6] There exists a partition of $\mathbf{R}^3$ into circles.*

Proof. It is easy to see that it is possible to partition a two-punctured sphere into disjoint circles. Let (x, y, z) be coordinates in $\mathbf{R}^3$ and consider the union C of the circles of radius 1 in the (x, y)-plane centered at the points $(4k + 1, 0, 0)$, $k \in \mathbf{Z}$. Then any sphere centered at the origin intersects C in exactly two points. By covering each such sphere with disjoint circles, we obtain the desired partition.

$\square$

In the construction above, every positive real number occurs infinitely often as the radius of a circle of the partition. Conway and Croft [3] claim to have shown that one could do the same thing with circles of the same radius, but their argument is somewhat unsatisfactory. We also refer to Kharazishvili [5]. Alternatively, one can demand that every positive real number should occur exactly once as the radius of a circle in the partition (see Theorem 1.1.6 below). The method to obtain such "strange" partitions relies on the Axiom of Choice and is general enough to deserve separate treatment.

We will consider the following problem. Let X be a set, and let S be a family of subsets of X. We ask whether it is possible to obtain a partition of X using only sets from S.

Definition 1.1.2 Let X and S be as above. The pair (X, S) is called *flexible* if given a subcollection $S' \subseteq S$ of disjoint sets with $|S'| < |X|$, and an $x \in X \setminus \bigcup S'$, we can always find an $s \in S$ such that $x \in s$ and $s \cap \bigcup S' = \emptyset$.

If (X, S) is flexible, we can sometimes think of S as consisting of "many but small" subsets of X. The usefulness of this notion comes from the following proposition.

Proposition 1.1.3 *If (X, S) is flexible then S has a subcollection which constitutes a partition of X.*

To prove this proposition, we will need a few set-theoretical concepts. A *well-ordering* of a set X is a linear ordering $<$ such that every nonempty subset of X has a minimal element. As an example, $\mathbf{N}$ is naturally well-ordered. It is a well known consequence of the Axiom of Choice that every set admits a well-ordering.

If y is an element of an ordered set X, then $X_{<y}$ will denote the set of all $x \in X$ such that $x < y$.

A well-ordering $<$ of a set X is said to be *minimal*, if for every $y \in X$, $|X_{<y}| < |X|$. For example, the usual ordering on $\mathbf{N}$ is a minimal well-ordering.

Lemma 1.1.4 *Every set X admits a minimal well-ordering.*

Proof. Let $<$ be a well-ordering on X. Let $Y = \{y \in X; |X_{<y}| < |X|\}$. We claim that $|Y| = |X|$. Suppose that this is not the case. Then $X \setminus Y$ contains a minimal element x. Since $|X_{<x}| = |Y| < |X|$, we have $x \in Y$, a contradiction. It follows that $Y = \{y \in X; |X_{<y}| < |Y|\} = \{y \in Y; |Y_{<y}| < |Y|\}$, i.e. that Y is minimally well-ordered. Now a bijection between X and Y induces a minimal well-ordering on X.

□

Proof of Proposition 1.1.3. Let $<$ be a minimal well-ordering of X. A subcollection S' of S is called *good* if:

- The sets in S' are pairwise disjoint.
- For every $x \in X$ and $s \in S'$, if $x < \min(s)$ then $x \in \bigcup S'$.

The good classes are partially ordered by inclusion and every ascending chain has a least upper bound, namely its union. By Zorn's lemma, there exists a maximal good subcollection S_0. We will show that S_0 is a partition of X. Clearly the elements of S_0 are disjoint subsets of X. Suppose that $X \setminus \bigcup S_0 \neq \emptyset$, and let x be its minimal element. By the second condition above, every element in S_0 must cover some element in $X_{<x}$. Since $<$ is minimal, we have $|S_0| \leq |X_{<x}| < |X|$. Now the flexibility of (X, S) implies the existence of an $s \in S$ such that $x \in s$ and $s \cap \bigcup S_0 = \emptyset$. But then $S_0 \cup \{s\}$ is good, which contradicts the maximality of S_0. Hence S_0 is a partition of X.

□

Our first application of this proposition is to prove the result of Conway-Croft and Kharazishvili.

Theorem 1.1.5 *[3], [5]. There exists a partition of* $\mathbf{R}^3$ *into congruent circles.*

Proof. We take $X = \mathbf{R}^3$ and let S be the collection of circles of unit radius. By Proposition 1.1.3 we only have to show that (X, S) is flexible. This will follow from Lemma 1.1.7 below.

□

Another application is the following:

Theorem 1.1.6 *There exists a partition of* $\mathbf{R}^3$ *into circles such that every positive real number appears exactly once as the radius of a circle in the partition.*

Proof. We take $X = \mathbf{R}^3 \sqcup \mathbf{R}^+$ (the disjoint union of $\mathbf{R}^3$ and $\mathbf{R}^+$) and let S be the collection of sets of the form $s \cup \{r\}$ where s is a circle in $\mathbf{R}^3$ with radius r. Once again Proposition 1.1.3 tells us that it is sufficient to prove that (X, S) is flexible. This too follows from Lemma 1.1.7.

□

Lemma 1.1.7 *Given any collection $\{s_\alpha\}_{\alpha \in A}$, $|A| < c$, of disjoint circles in $\mathbf{R}^3$, there exists a circle s in $\mathbf{R}^3$, disjoint from all the s_α's, which passes through an arbitrarily prescribed point in $\mathbf{R}^3$ (not covered by the s_α's), and has an arbitrarily prescribed positive radius.*

Proof. We will construct a new circle which passes through a given point x (not in the union of the s_α's) and has a given radius $r > 0$. Each circle s_α lies in exactly one plane in $\mathbf{R}^3$. Since there are fewer than c circles, we can find a plane through x intersecting each circle s_α at most twice. In this plane, we can draw c different circles of radius r through x, and each s_α intersects at most 4 of these. Clearly, we can pick one which is disjoint from all the s_α's.

□

We want to stress that Proposition 1.1.3 is a very powerful tool for constructing partitions. It allows us to establish the existence of partitions of $\mathbf{R}^3$ into circles satisfying seemingly very strong conditions. The following theorem is an example of this. We let $\mathbf{Q}^+$ be the set of positive rational numbers and let Π be the set of planes in $\mathbf{R}^3$.

Theorem 1.1.8 *Let Y be a subset of $\mathbf{R}^3$ with cardinality less than c (e.g. the set of points with rational coordinates). Then there exists a partition of $\mathbf{R}^3 \setminus Y$ into circles such that:*

- *Every positive real number appears exactly once as the radius of a circle in the partition.*
- *Every plane in $\mathbf{R}^3$ contains exactly one circle of the partition.*
- *Every point in $\mathbf{R}^3$ is the center of exactly one circle of the partition.*
- *Given any rational number $q > 0$, the collection of circles obtained from the original partition by magnifying each circle by a factor q (keeping its center and plane fixed) constitutes a new partition of $\mathbf{R}^3 \setminus Y$.*

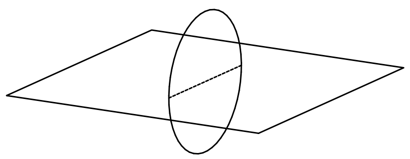

Figure 1.1: The plane $z = 0$ and a circle which is "admissible" in the proof of Theorem 1.2.1.

Proof. We use Proposition 1.1.3 with $X = ((\mathbf{R}^3 \setminus Y) \times \mathbf{Q}^+) \sqcup \mathbf{R}^+ \sqcup \Pi \sqcup \mathbf{R}^3$ and let S be the collection of subsets of the form $\bigcup_{q \in \mathbf{Q}^+} (s_q, q) \cup \{r\} \cup \{\pi\} \cup \{p\}$ where $s = s_1$ is a circle in $\mathbf{R}^3$, r, π and p are the radius, plane and center associated with s, and s_q is the circle in $\mathbf{R}^3$ obtained by magnifying s by a factor q, keeping its plane π and center p fixed. We claim that (X, S) is flexible. This can be proved by an argument similar to Lemma 1.1.7. The statement of the lemma will now be more complicated but the proof very much the same. We omit the details.

$\square$

1.2 Unlinked circles

Szulkin's construction uses linked circles, but this is, as we shall see, not necessary even if we demand that all circles be congruent. This answers a question posed in [7]. Here we present two results concerning partitions with unlinked circles.

Theorem 1.2.1 *There exists a partition of $\mathbf{R}^3$ into unlinked circles such that every positive real number occurs exactly once as a radius.*

In the proof of this theorem, we will only consider circles in $\mathbf{R}^3$ of a special kind. Let (x, y, z) be coordinates in $\mathbf{R}^3$. A circle in $\mathbf{R}^3$ is called *admissible* if it is symmetric with respect to the plane $z = 0$ (see Figure 1.1).

We note that two disjoint admissible circles cannot be linked. The analogue of Lemma 1.1.7 is the following result.

Lemma 1.2.2 *Given any collection $\{s_\alpha\}_{\alpha\in A}$, $|A| < c$, of disjoint admissible circles, we can find an admissible circle s, disjoint from all the s_α's, for which we can prescribe either that it should pass through a given point in $\mathbf{R}^3$ (not covered by the s_α's) and have any sufficiently large radius, or that it should have a given positive radius.*

Proof. We show that, under the above conditions, we can cover any point $p = (x, y, z)$ (not covered by the s_α's) with an admissible circle of a given radius $r > |z|$. This will take care of both cases of the lemma.

For each s_α there are at most two admissible circles with radius r through p intersecting s_α. There are c admissible circles with radius r through p and the s_α's are fewer than c in number, so there has to be an admissible circle with radius r passing through p which is disjoint from all the s_α's.

$\square$

Proof of Theorem 1.2.1. We apply Proposition 1.1.3 with $X = \mathbf{R}^3 \sqcup \mathbf{R}^+$ and let S be the collection of sets of the form $s \cup \{r\}$, where s is an admissible circle with radius r. By Lemma 1.2.2, (X, S) is flexible. Since we only consider admissible circles, we get a partition of $\mathbf{R}^3$ into unlinked circles.

$\square$

Next we show that the circles in a partition do not have to be linked even if we require that they have the same radius.

Theorem 1.2.3 *There exists a partition of $\mathbf{R}^3$ into unlinked circles of unit radius.*

Proof. In order to prove this, we have to change our class of admissible circles. From now on a circle is called *admissible* if it has radius 1, and is symmetric with respect to one of the planes $z = -1$ or $z = 1$ (see Figure 1.2).

Note that if we can cover all points with z-coordinate strictly between -2 and 2 with disjoint admissible circles, then exactly one of the points $(x, y, -2)$ and $(x, y, 2)$ will be covered. Hence this covering, together with all its vertical translates by the numbers $4k$, $k \in \mathbf{Z}$, will constitute a partition of $\mathbf{R}^3$. Clearly, two circles used in this construction cannot be linked.

We use Proposition 1.1.3 with $X = \mathbf{R}^3$, and let S be the class of all sets consisting of an admissible circle, together with all its vertical translates by the numbers $4k$, $k \in \mathbf{Z}$. The flexibility of (X, S) is guaranteed by the following lemma.

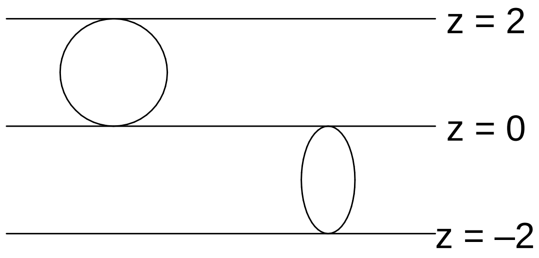

Figure 1.2: Circles which are "admissible" in the proof of Theorem 1.2.3.

□

Lemma 1.2.4 *Given a collection* $\{s_\alpha\}_{\alpha\in A}$ *of disjoint, admissible circles,* $|A| < c$*, and a point* (x, y, z)*, with* $-2 < z < 2$*, not covered by the* s_α*'s, there exists an admissible circle passing through* (x, y, z) *which is disjoint from all the* s_α*'s.*

Sketch of Proof. There are always c admissible circles passing through the point (x, y, z), and each s_α can intersect at most finitely many of these.

□

1.3 Partitioning $\mathbf{R}^3$ into other curves

So far, all the generalizations of Theorem 1.1.1 have consisted of putting seemingly restrictive conditions on the circles of the partition. Now we generalize in another direction, replacing the circles by more general curves. It is not difficult to use the ideas above to partition $\mathbf{R}^3$ into *plane* curves, e.g. ellipses or polygons, but to use other curves we need a refinement of the technique.

By a *real analytic curve* in $\mathbf{R}^3$ we mean a smooth embedded curve which is locally parametrized by a nonsingular real analytic function.

Theorem 1.3.1 *Let* $\{\gamma_\alpha\}_{\alpha\in A}$*,* $|A| = c$ *be a collection of real analytic curves in* $\mathbf{R}^3$*. Then there exist orientation preserving isometries* $\{g_\alpha\}$ *of* $\mathbf{R}^3$ *such that* $\{g_\alpha\gamma_\alpha\}$ *is a partition of* $\mathbf{R}^3$*.*

Again we will use Proposition 1.1.3. This time we take $X = \mathbf{R}^3 \sqcup A$, and let $S = \{g\gamma_\alpha \cup \{\alpha\}\}$, where g is an orientation preserving isometry of $\mathbf{R}^3$, and $\alpha \in A$. The following lemma implies that (X, S) is flexible.

Lemma 1.3.2 *Given a collection* $\{\gamma_\alpha\}$ *of fewer than c real analytic curves in* $\mathbf{R}^3$*, a point* $p \in \mathbf{R}^3 \setminus \bigcup \gamma_\alpha$ *and another real analytic curve* γ*, there exists an orientation preserving isometry* g *of* $\mathbf{R}^3$ *such that* $p \in g\gamma$ *and* $g\gamma$ *is disjoint from all the* γ_α*'s.*

Proof. Without loss of generality we may assume that $p \in \gamma$ and that $p = 0$. It suffices to find a rotation g around the origin such that $g\gamma$ does not intersect any of the γ_α's. We restrict our attention to the set T of rotations by an angle less than $\pi/2$, with fixed points in the (x, y)-plane. We can identify T with the upper half of the unit sphere by representing the rotation g by the point $g(0, 0, 1)$.

We will use the fact that a real analytic curve cannot intersect a sphere in more than a countable number of points, unless it lies entirely on the sphere. Now γ contains the origin, hence does not lie on a sphere centered at the origin. The curve γ has a real analytic parametrization $\gamma(t) = (x(t), y(t), z(t))$. Let $r_\gamma(t) = \sqrt{x^2 + y^2 + z^2}$. We can divide γ into a countable number of pieces such that every piece is either a point or an arc on which $dr_\gamma(t)/dt \neq 0$.

Let l_1 be one of these pieces. Let $\alpha \in A$. We want to show that the set of rotations $g \in T$ such that $g(l_1)$ intersects γ_α is represented by a countable union of real analytic curves on the sphere.

This is clear if either l_1 consists of a single point or if γ_α lies on a sphere centered at the origin.

We therefore assume that γ_α does not lie on a sphere centered at the origin. Hence we can divide γ_α into pieces in the same way as we did with γ.

Let l_2 be an arc (or a point) from γ_α. It suffices to show that the set of rotations $g \in T$ such that $g(l_1)$ intersects l_2 is represented by a real analytic curve on the sphere. This is clear if l_2 consists of a single point.

Suppose without loss of generality that $dr_\gamma(t)/dt > 0$ on l_1. Then by the Inverse Function Theorem (for real analytic functions), the inverse function $t = t(r)$ is real analytic. Hence l_1 has a real analytic parametrization $l_1(r) = (x_1(r), y_1(r), z_1(r))$, where r is the distance to the origin. Similarly, l_2 has a real analytic parametrization $l_2(r) = (x_2(r), y_2(r), z_2(r))$. Consider an interval where both $l_1(r)$ and $l_2(r)$ are defined, and the angle between them is less than $\pi/2$. For every r in this interval, there is a unique rotation $g_r \in T$ such that $g_r(l_1(r)) = l_2(r)$. By geometric considerations, we see that it is possible to find an expression for $g_r(0,0,1)$ in terms of $l_1(r)$ and $l_2(r)$, involving only rational functions and square roots. Hence $g_r(0,0,1)$ is a real analytic function of $l_1(r)$ and $l_2(r)$, and it follows that the composite function $r \mapsto g_r(0,0,1)$ is real analytic.

In other words, the set $E \subseteq T$ of rotations g such that $g(l_1(r)) = l_2(r)$ for some r in this interval is represented by a real analytic curve on the upper half of the unit sphere.

It now follows that the set of g such that $g\gamma$ intersects some γ_α is represented by a union of fewer than c real analytic curves on the upper hemisphere. We choose a circle on the upper hemisphere which does not contain any of these curves. Since this circle intersects each of the curves in only a countable number of points, it must contain a point which does not lie on any of these curves. This point represents the desired rotation g, and the proof is complete.

□

Remark 1.3.3 Using the same technique, we can show that Theorem 1.3.1 remains true even if we allow the sets in A to be unions of fewer than c real analytic curves (for example polygons).

Remark 1.3.4 Lemma 1.3.2 above (and hence the theorem) should probably be true under considerably weaker smoothness assumptions on the curves, say C^1 (perhaps the Continuum Hypothesis would help here).

One must, however, assume some regularity, i.e. one cannot replace the family γ_α in the theorem by any family of Jordan curves. To see this, we use the fact that a Jordan curve in $\mathbf{R}^3$ can have positive Lebesgue measure. More precisely, given $\epsilon > 0$ we can, by modifying a construction in [4], find a Jordan curve γ in the cube $E = [0,1]^3$, not passing through $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, with Lebesgue measure $\lambda(\gamma) > 1 - \epsilon$.

We claim that $\mathbf{R}^3$ cannot be partitioned into isometric copies of γ if ϵ is small enough. It suffices to show that if γ' is an isometric copy of γ passing through $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, then γ and γ' must intersect. But for simple geometrical reasons there exists an $\alpha > 0$, not depending on ϵ, such that the cubes E_γ and $E_{\gamma'}$ corresponding to γ and γ' must intersect in a set of measure at least α. If γ and γ' were disjoint, then we would have $2 - \alpha \geq 2 - \lambda(E_\gamma \cap E_{\gamma'}) = \lambda(E_\gamma \cup E_{\gamma'}) \geq \lambda(\gamma \cup \gamma') = \lambda(\gamma) + \lambda(\gamma') \geq 2 - 2\epsilon$. This is a contradiction if $2\epsilon < \alpha$.

1.4 Higher dimensions

The method of Section 1.1 generalizes to prove the following results (see the proof of Theorem 1.4.5):

Theorem 1.4.1 *[2] There exists a partition of* $\mathbf{R}^{2n+1}$ *into (isometric) copies of* S^n*.*

Using the methods of Section 1.2 one can even demand that the n-spheres be unlinked. The following question is natural:

Question 1.4.2 *What is the smallest* $m = m(n)$ *such that* $\mathbf{R}^m$ *can be partitioned into* n*-spheres?*

Theorem 1.4.1 shows that $m(n) \leq 2n+1$. On the other hand, one easily sees that $m(n) \geq n+2$ (see Theorem 1.4.9 below).

Partitioning an open subset of $\mathbf{R}^3$ (or $\mathbf{R}^{2n+1}$) is no more difficult than the whole space:

Theorem 1.4.3 *[5] Any open subset of* $\mathbf{R}^3$ *(or* $\mathbf{R}^{2n+1}$*) can be partitioned into circles (or* n*-spheres).*

Another natural issue is to partition $\mathbf{R}^{2n+1}$ into subsets of S^n:

Theorem 1.4.4 *[2] Given a nonempty subset* $U \subset S^n$*, there exists a partition of* $\mathbf{R}^{2n+1}$ *into (isometric) copies of* U*.*

As an example of the technique, we give a proof of the following theorem.

Theorem 1.4.5 *Given a nonempty subset* $U \subset \mathbf{R}^n$*, there exists a partition of* $\mathbf{R}^{2n+1}$ *into (isometric) copies of* U*.*

Proof. We show that the copies of U constitute a flexible family. It is enough to show that, given a collection $\{V_\alpha\}$ of fewer than c n-dimensional affine subspaces in $\mathbf{R}^{2n+1}$, we can find another n-dimensional affine subspace V, containing a given point (not in the union of the V_α's), say the origin, without intersecting any V_α.

Let $y_{\alpha,1}, \ldots, y_{\alpha,n+1}$ be an affine basis for V_α. Since we want to cover the origin, we want to find points $x_1, \ldots, x_n$ in $\mathbf{R}^{2n+1}$ whose (linear) span does not intersect any V_α, or equivalently, such that

$$\det(y_{\alpha,1}, \ldots, y_{\alpha,n+1}, x_1, \ldots, x_n) \neq 0$$

for all α. Since this determinant is a polynomial in the n variables $x_1, \ldots, x_n$, in other words, in $n(2n+1)$ real variables, the theorem is a consequence of the following lemma.

□

Lemma 1.4.6 *Given fewer than c nonzero polynomials P_α in m real variables, there exists a point in $\mathbf{R}^m$ at which none of them vanishes.*

Proof. Since a polynomial can have only a finite number of linear factors, each polynomial vanishes on only finitely many hyperplanes. Since there are c hyperplanes in $\mathbf{R}^m$, there is a hyperplane H on which no P_α is identically zero. By a linear substitution of variables, the restrictions of the P_α's to H can be regarded as polynomials in $m-1$ variables. Hence the lemma will follow by induction on m.

□

We can ask whether the last theorem is sharp, i.e whether there exists a subset $U \subset \mathbf{R}^n$ such that $\mathbf{R}^{2n}$ cannot be partitioned into isometric copies of U. Obviously, $\mathbf{R}^{2n-1}$ cannot be partitioned into copies of $\mathbf{R}^n \setminus \{0\}$, since it is impossible to use one copy of $\mathbf{R}^n \setminus \{0\}$ to fill the "hole" of another, unless they intersect.

Question 1.4.7 *Is it possible to partition $\mathbf{R}^{2n}$ into copies of $\mathbf{R}^n \setminus \{0\}$?*

We remark that Question 1.4.2 has been answered if we allow homeomorphic images of spheres. On the one hand, one can indeed do much better than the estimate $m(n) \leq 2n+1$ suggests:

Theorem 1.4.8 *[1] There exists a partition of $\mathbf{R}^{n+2}$ into homeomorphic images of S^n.*

On the other hand, we have the following well-known fact:

Theorem 1.4.9 $\mathbf{R}^{n+1}$ *cannot be partitioned into homeomorphic images of S^n.*

This is perhaps most easily proved using the Hausdorff Maximality Principle, but one can give a more constructive argument which, as far as the authors know, has not appeared before: *Proof.* Suppose a partition exists and let s be one of the sets of the partition. Then s divides $\mathbf{R}^{n+1}$ into two parts, one of which, U, is bounded (the Jordan Curve Theorem). Take any $x_0 \in U$ with maximal distance to the boundary and let U_0 be the bounded component of the complement of the set of the partition passing through x_0. Define inductively x_k to be any element in U_{k-1} with maximal distance to the boundary and let U_k be the bounded component of the complement of the element passing through x_k. Let r_k be the distance from x_k to the boundary of U_{k-1}. An easy compactness argument shows that $r_k \to 0$ as $k \to \infty$. The sets U_k form a decreasing sequence of open subsets of U. Let y be any cluster point of $\{x_k\}$. Then y belongs to all the U_k and the maximal distance from y to the boundary of U_k tends to 0. But then there can be no set of the partition passing through y.

□

1.5 Bijection spaces

We end this paper by a different application of our method.

Theorem 1.5.1 *Given a bijection f of $\mathbf{R}^2$ onto itself, we can construct a one-parameter family of bijections $f_t : \mathbf{R}^2 \to \mathbf{R}^2$ such that*

- $f_0 = id$
- $f_1 = f$
- *for every point x in $\mathbf{R}^2$, the function $g : [0,1] \to \mathbf{R}^2$ given by $g(t) = f_t(x)$, is continuous.*

Proof. Let $X = [0,1] \times \mathbf{R}^2$, and let S be the set of all graphs of continuous functions $g : [0,1] \to \mathbf{R}^2$ with the property that, for some $x \in \mathbf{R}^2$, $g(0) = x$ and $g(1) = f(x)$. The theorem is equivalent to the statement that there is a partition of X into sets from S. By Remark 1.3.4, Section 1.3, it seems unlikely that (X, S) is flexible. Therefore, we restrict ourselves to the subclass $S' \subseteq S$ consisting of all graphs of piecewise linear functions g. We now show that (X, S') is flexible.

Suppose that we are given a collection $\{g_\alpha\}$ of fewer than c sets from S', and a point (t, y) in X, not covered by any g_α. If $0 < t < 1$, we choose a point $x \in \mathbf{R}^2$ such that $(0, x)$ is not covered by any g_α. Then neither is $(1, f(x))$. If $t = 0$ or $t = 1$, we let $x = y$ or $x = f^{-1}(y)$, respectively. We want to find a piecewise linear function g whose graph passes through the points $(0, x)$, (t, y) and $(1, f(x))$ without intersecting any g_α.

If $t \neq 0$, consider the planes containing the points $(0, x)$ and (t, y). If a line segment from a g_α does not lie on the line joining $(0, x)$ to (t, y), it lies in at most one of these planes. Hence we can find such a plane which, except perhaps on the line through $(0, x)$ and (t, y), intersects the g_α's in fewer than c points. In this plane, we can then find a point (t', z), $0 < t' < t$, such that the line segments joining $(0, x)$ to (t', z), and joining (t', z) to (t, y) will not intersect any of the g_α's. By joining (t, y) to $(1, f(x))$ in the same way, we obtain the desired function g.

□

On a bounded set, we can even make the points move in a uniformly continuous way.

Theorem 1.5.2 *Let U be the open unit disk. Let f be a bijection $U \to U$. Then there are bijections $f_t : U \to U$ such that*

- $f_0 = id$
- $f_1 = f$
- *for* $s, t \in [0, 1]$ *and* $x \in U$, $|f_s(x) - f_t(x)| \leq 4\,|s - t|$.

Proof. We will partition the set $X = [0, 1] \times U$ into graphs of piecewise linear functions with slope ≤ 4. We proceed as in the proof of the previous theorem. Given a collection $\{g_\alpha\}$ of fewer than c such graphs, and a point $(t, y) \in [0, 1] \times U$, not in their union, we can choose the point x such that either x or $f(x)$ is as close as we please to y, depending on whether t is closer to 0 or 1. Since the distance between any two points in U is less than 2, the line segments joining $(0, x)$ to (t, y), and joining (t, y) to $(1, f(x))$ will both have slope strictly less than $\frac{2}{1/2} = 4$. As in the proof of the previous theorem, we now choose a point (t', z), and join $(0, x)$ to (t, y) via (t', z). Since the point (t', z) can be chosen arbitrarily close to the straight line from $(0, x)$ to (t, y), this can be done in such a way that all the line segments used have slope ≤ 4.

□

Let (X, ρ) be a bounded metric space. We define the *bijection space* of (X, ρ) to be the set of all bijections of X onto itself, with the metric given by $d(f, g) = sup_{x \in X} \rho(f(x), g(x))$. Theorem 1.5.2 can then be reformulated as follows:

The bijection space of the open unit disk in $\mathbf{R}^2$ is path connected.

Remark 1.5.3 It follows from well known theorems of algebraic topology that the set of all continuous bijections is *not* path connected in this metric. There is no way to get from the identity to a reflection in a line via continuous bijections.

Question 1.5.4 *Which metric spaces give rise to path connected bijection spaces?*

For example it is easily verified, using the above method, that two disks together with a common boundary point, or even two disks connected by a line segment, give rise to path connected bijection spaces. Hence a gas can pass through a hole consisting of only one point! We can also ask other questions about the topology of the bijection space.

Theorem 1.5.5 *The bijection space of the open unit disk is not simply connected.*

Proof. Consider the closed curve in the bijection space consisting of rotations around the origin by an angle $\theta \in [0, 2\pi]$. Suppose that this curve could be contracted to the identity. Without loss of generality, we can assume that the origin remains fixed throughout the contraction. Then the path of any other point x, which is a circle around the origin, must be contracted to the point x without passing over the origin. This is impossible.

$\square$

Question 1.5.6 *Is the bijection space of the three dimensional open ball simply connected?*

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