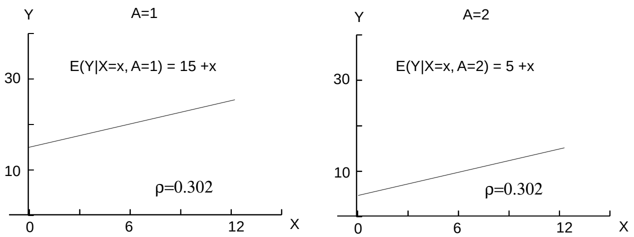
Effect reversal

or: why it is important to study relations among explanatory variables

- Example for variables Y, X, A
- Example for variables A, B, C
- Example for variables Y, X, Z
- A general condition when it cannot occur

Slide 1

Example of reversal of the effect of X on Y



Here X is strongly dependent on A since

$\mathrm{E}(X \mid A = 1) = 5$ and $\mathrm{E}(X \mid A = 2) = 11$

and has little variability given A: $\mathrm{var}(X \mid A = 1) = \mathrm{var}(X \mid A = 2) = 1$

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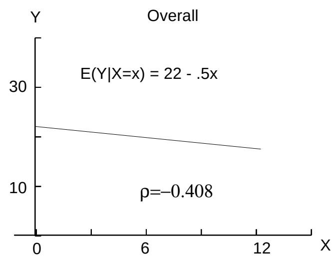
As a consequence of strongly associated explanatory variables X, A
a reversal in the effect of X on Y occurs, when A is ignored



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Details to the example

Levels	Means		Variances			
of A	Y	X	Y	X	$\sigma_{X,Y}$	$P(A=i)$
$i=1$	20	5	11	1	1	.5
$i=2$	16	11	11	1	1	.5
Overall	18	8	15	10	-5	

with e.g.

$$\begin{aligned}\sigma_{XY} &= E_A\{\mathrm{cov}(Y, X \mid A)\} + \mathrm{cov}_A\{E(Y \mid A), E(X \mid A)\} \\ &= (.5 \times 1 + .5 \times 1) \\ &\quad +\{.5 \times (20-18)(5-8) + .5 \times (16-18)(11-8)\} \\ &= -5\end{aligned}$$

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In two clinics a new treatment $(j = 1)$ is better, but....

	C, Clinic $k = 1$		C, Clinic $k = 2$		counts for AB ignoring C	
A, Treatment success	B, Treatment $j = 1$ new	B, Treatment $j = 2$ conv.	B, Treatment $j = 1$ new	B, Treatment $j = 2$ conv.	B, Treatment $j = 1$ new	B, Treatment $j = 2$ conv.
$i = 1$: yes	60	4	30	200	90	204
$(P_{1\mid jk})$	(30%)	(20%)	(75%)	(50%)	(38%)	(49%)
$i = 2$: no	140	16	10	200	150	216
sum	200	20	40	400	240	420
rel. chance	$30/20 = 1,5$		$75/50 = 1,5$		$38/49 = 0,78$	

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Replicated relative chances are preserved since B and C are not associated

	C, Clinic $k = 1$		C, Clinic $k = 2$		counts from both clinics	
A, Treatment success	B, Treatment $j = 1$ new	B, Treatment $j = 2$ conv.	B, Treatment $j = 1$ new	B, Treatment $j = 2$ conv.	B, Treatment $j = 1$ new	B, Treatment $j = 2$ conv.
$i = 1$: yes	6	40	30	200	36	240
$(P_{1\mid jk})$	(30%)	(20%)	(75%)	(50%)	(60%)	(40%)
$i = 2$: no	14	160	10	200	24	360
sum	20	200	40	400	60	600
rel. chance	$30/20 = 1,5$		$75/50 = 1,5$		$60/40 = 1,5$	

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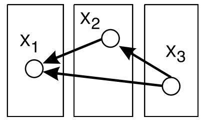
Overall effect in a Gaussian linear system

For mean-centered variables X_1, X_2, X_3 the **triangular system** with the graph



corresponding linear equations can be written as:

$$\begin{aligned} \mathrm{E}(X_1 \mid X_2, X_3) &= \beta_{1|2.3}X_2 + \beta_{1|3.2}X_3 \\ \mathrm{E}(X_2 \mid X_3) &= \beta_{2|3}X_3 \\ \mathrm{E}(X_3) &= 0 \end{aligned}$$

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The **overall effect** of X_3 on X_1 is the sum of effects of 'two paths' since

$$\begin{aligned} \mathrm{E}(X_1 \mid X_3) &= \mathrm{E}_{X_2|X_3}\mathrm{E}(X_1 \mid X_2, X_3) \\ &= \mathrm{E}_{X_2|X_3}(\beta_{1|2.3}X_2 + \beta_{1|3.2}X_3) \\ &= \beta_{1|2.3}\mathrm{E}_{X_2|X_3}(X_2) + \beta_{1|3.2}X_3 \\ &= \beta_{1|2.3}\beta_{2|3}X_3 + \beta_{1|3.2}X_3 \\ \text{or} & \\ \beta_{1|3}X_3 &= (\beta_{1|2.3}\beta_{2|3} + \beta_{1|3.2})X_3 \end{aligned}$$

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Suppose a Response , Treatment, and Backround variable are standardized to have mean zero and variance 1; they be linearly related and correlated as

$$\mathrm{cor} = \begin{pmatrix} 1 & -.32 & .544 \\ . & 1 & -.8 \\ . & . & 1 \end{pmatrix}$$

then - by computing least squares regression coefficents - we get

$$\begin{aligned} \mathrm{E_{lin}(R \mid T, B)} &= .32 \times \mathrm{T} + .8 \times \mathrm{B} \\ \mathrm{E_{lin}(T \mid B)} &= -.8 \times \mathrm{B} \end{aligned}$$

and $\beta_{\mathrm{RT}} = -.32$, i.e. effect reversal

But for the same $\mathrm{E_{lin}(R \mid T, B)}$ with T, B uncorrelated

$$\begin{aligned} \mathrm{E_{lin}(R \mid T, B)} &= .32 \times \mathrm{T} + .8 \times \mathrm{B} \\ \mathrm{E_{lin}(T \mid B)} &= 0 \times \mathrm{B} \end{aligned}$$

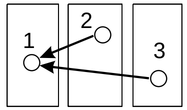
$\beta_{\mathrm{RT}} = .32$, i.e. effect is preserved

The correlation matrix is

$$\mathrm{cor} = \begin{pmatrix} 1 & .32 & .8 \\ . & 1 & 0 \\ . & . & 1 \end{pmatrix}$$

On the overall effect for general distributions

generated over the graph



The joint density is in condensed notation

$$f_{123} = f_{1|23} f_2 f_3$$

The overall effect of variable 3 on variable 1 is the dependence in

$$f_{1|2} = \int f_{1|23} f_3 dx_3$$

$X_2 \perp\!\!\!\perp X_3$ is sufficient for no effect reversal in the case variable X_1 depends monotonically on X_2 for all (possible reordered) levels of X_3 (Cox and Wermuth, 2003)

This means that at least qualitatively one obtains the same conclusions regarding the direction of dependence of variable 1 on variable 2, no matter whether variable 3 is explicitly considered (with $f_{1|23}$) or not (with $f_{1|2}$)

Without monotonous dependence - as e.g. in the Lienert data - no such conclusions are possible