Measures everywhere

High intensity optimisation

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$\mathcal{L}_1$-Approximation of convex functions

Let $f(y)$ be a convex function on $[a,b]$.

$g(y,P)$ – linear spline: $g(y_i,P) = f(y_i)$ for $y_i \in P$ – set of N points in $[a,b]$.

Problem 0: to improve accuracy of the trapezoidal rule, i. e.

find P that minimises

$$F(P) = \int_a^b [g(y,P) - f(y)]\, dy\,.$$

Asymptotic solution when N grows is given by McClure & Vitale (75): density of P should be proportional to $f''(x)^{1/3}$.

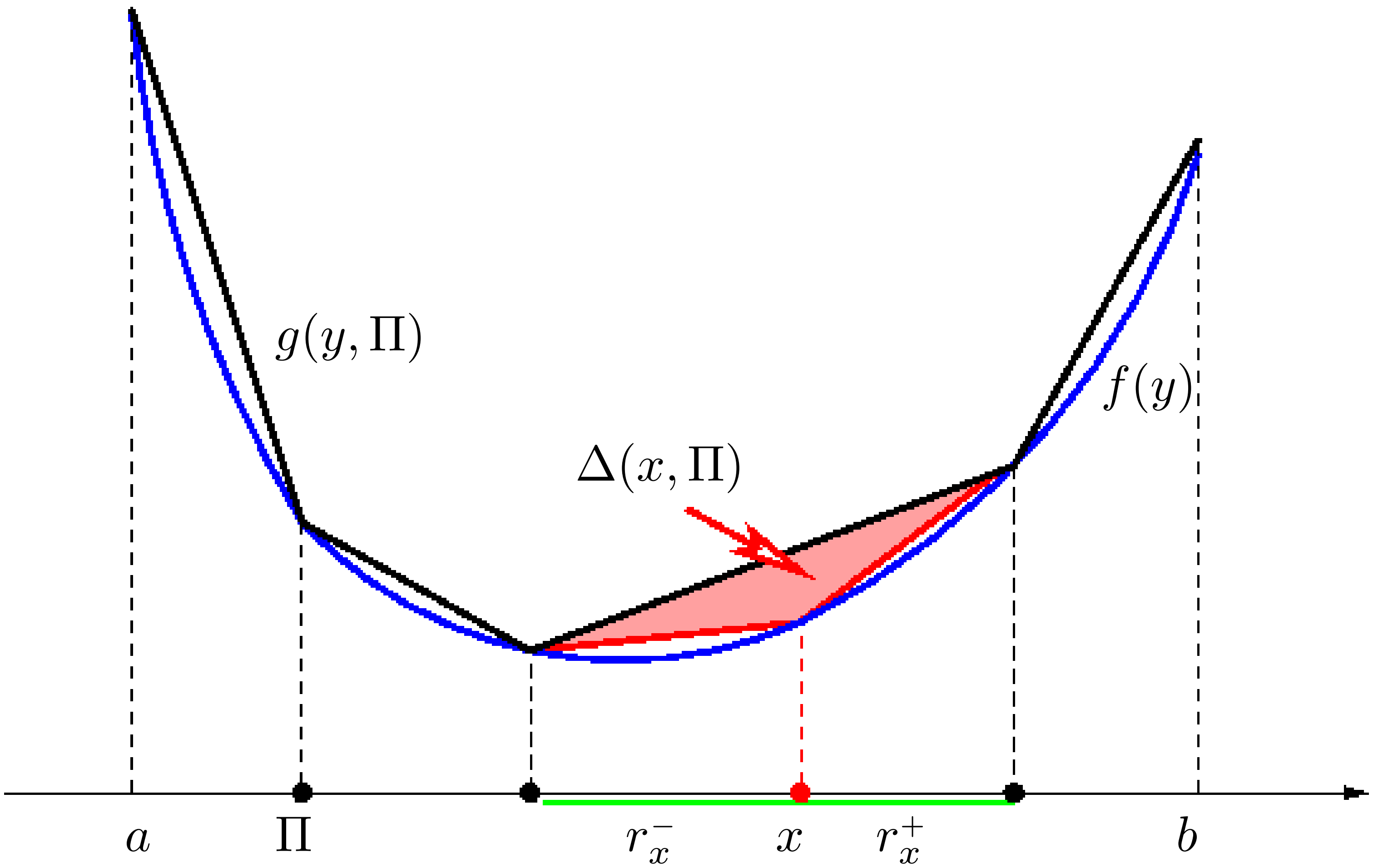
Poissonisation

Let P be random and consider

Problem: find μ minimising expectation

$$\mathbf{E}_\mu F(\Pi) = \int_a^b g(y, \Pi) - f(y)\, dy\,.$$

subject to $\mu([a,b]) = N$ – large but fixed.



$$\Delta(x,\Pi) = F(\Pi + \delta_x) - F(\Pi) = \text{Area}(\text{Triangle}),$$

Using independence of r_x^-, r_x^+ we obtain:

$$\overline{\Delta}_\mu(x) = -f(x)[\mathbf{E}_\mu r_x^- + \mathbf{E}_\mu r_x^+] + \mathbf{E}_\mu r_x^- \; \mathbf{E}_\mu f(x + r_x^+)$$
$$+ \mathbf{E}_\mu r_x^+ \; \mathbf{E}_\mu f(x - r_x^-) = \mathrm{Const} \qquad (1)$$

Noting that

$$\mathbf{P}_\mu\{r_x^- > t\} = \exp\{-\mu([x - t, x])\}$$
$$\mathbf{P}_\mu\{r_x^+ > t\} = \exp\{-\mu([x, x + t])\}$$

it can first be shown that the density $p_N(x)$ of an optimal μ_N with $\mu([a, b]) = N$ exists and that (1) leads to a DE to be solved for each particular f.

High intensity solution

Assume $a\mu_a$ is the optimal measure with total mass a, so that $\mu_a(X) = 1$. How does μ_a behave for large a? We say that μ is a *high-intensity solution* if μ_a tends to μ in some sense (e.g., weakly).

Assume that all μ_a have densities $p_a(x) = \frac{d\mu_a}{d\ell}(x)$ and that for $x \in \mathrm{Int}X$ one has

$$\lim_{\substack{y \to x \\ a \to \infty}} p_a(y) = p(x) > 0\,.$$

$r_x^\pm$ have order $O(1/a)$ (with exp. tails), so that $p_a(x)$ is practically a constant $p(x)$ in such a small neighbourhood. Thus $r_x^\pm$ asymptotically conform to $\mathrm{Exp}(ap(x))$ – distance to the closest point in *homogeneous* Poisson process with intensity $ap(x)$.

Then, denoting $\mathbf{E}$ the expectation w.r.t. $\mathrm{Exp}(ap(x))$ and writing Taylor series for $f(x \pm r_x^\pm)$, (1) gives

$$\begin{aligned}\mathrm{Const} = \overline{\Delta}_\mu\,(x) &= -f(x)[\mathbf{E}\,r_x^- + \mathbf{E}\,r_x^+] + \mathbf{E}\,r_x^-\ \mathbf{E}\,f(x + r_x^+)\\ &+ \mathbf{E}\,r_x^+\ \mathbf{E}\,f(x - r_x^-) = a^{-3}p(x)^{-3}f''(x) + o(a^{-3})\end{aligned}$$

so that

$$p(x) \propto f''(x)^{1/3}$$

❑ Smth. to check here! It works because the influence of adding a point is local: $\overline{\Delta}_\mu\,(x)$ depends only on a *stopping set* $[x - r_x^-, x + r_x^+]$ that 'shrinks' as a grows.

Generalisations

- L_β-norm. Then $p(x) \propto (f''(x))^{\beta/(2\beta+1)}$

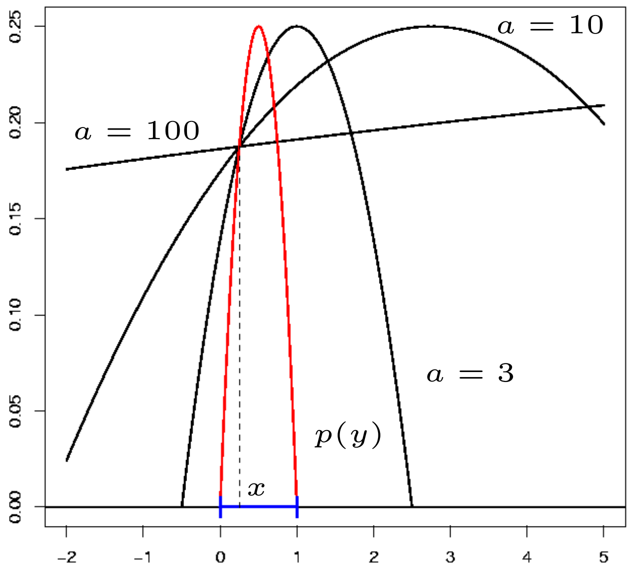
- L_1 approximation of d-dimensional function by maxima of tangent planes drawn at Poisson points $p(x) \propto K(x)^{1/(2+d)}$, $K(x) = \det \|D^2 f(x)\|$ – the Gaussian curvature of the surface.

- approximation of smooth convex sets by inscribed polygons (area) $p(x) \propto k(x)^{2/3}$

Existence of high intensity solution

Fix x and consider homothety γ_a^x with coefficient $a^{1/d}$:

$y \mapsto x + a^{1/d}(y - x)$



If we define for $\Pi = \sum \delta_{x_i}$

$\gamma_a^x \Pi \stackrel{\text{def}}{=} \sum \delta_{\gamma_{a^{-1}}^x x_i}$

then $\mathbf{E}_{a\mu_a} F(\Pi) = \mathbf{E}_{\hat{\mu}_a^x} F(\gamma_a^x \Pi)$

where $\hat{\mu}_a^x(\cdot) = a\mu_a(\gamma_{a^{-1}}^x \cdot)$ is concentrated on $\gamma_a^x X$.

Assume:

$$x \in \mathrm{Int}X, \quad p_a = \frac{d\mu_a}{d\ell}, \quad \lim_{\substack{y \to x \\ a \to \infty}} p_a(y) = p(x) > 0$$

Path differentiability:

For some $g(a) = g(a, x) > 0$

$$\Gamma_a(x;\Pi) = g^{-1}(a)\Delta(x;\gamma_a^x\Pi) \xrightarrow{\mathbf{P}_\ell-\text{a.s.}} \Gamma(x;\Pi) \text{ as } a \to \infty$$

such that $0 < \mathbf{E}_{p(x)\ell}\,\Gamma(x;\Pi) < \infty$.

Localisation:

There exist stopping sets $S_a = S_a(x;\Pi)$ and $S = S(x;\Pi)$ such that $\Gamma_a(x;\Pi)$ is $\mathcal{F}_{S_a}$-measurable $\forall a \geq A$; $\Gamma(x;\Pi)$ is $\mathcal{F}_S$-measurable; and for each compact set W containing x in its interior

$$1\!\!1_{S_a(x;\Pi)\subseteq W} \xrightarrow{\mathbf{P}_\ell-\text{a.s.}} 1\!\!1_{S(x;\Pi)\subseteq W} \text{ as } a \to \infty\,.$$

Uniform integrability:

There exists a compact set W containing x in its interior such that

$$\lim_{\substack{a\to\infty \\ b\to\infty}} \mathbf{E}_{\hat{\mu}_a^x} \left|\Gamma_a(x;\Pi)\right| \mathbb{1}_{S_a \not\subseteq \gamma_b^x W} = 0\,.$$

There exists a constant $M = M(W,b)$ such that $\left|\Gamma_a(x;\Pi)\right| \le M$ for all $a \ge A$ and Π satisfying $S_a(x;\Pi) \subseteq \gamma_b^x W$.

Then

$$\lim_{a\to\infty} \left| \mathbf{E}_{\hat{\mu}_a^x}\, \Gamma_a(x;\Pi) - \mathbf{E}_{p(x)\ell}\, \Gamma(x;\Pi) \right| = 0\,,$$

$$\lim_{a\to\infty} \frac{\overline{\Delta}_{a\mu_a}(x)}{\overline{\Delta}_{ap(x)\ell}(x)} = 1$$

and for ℓ-almost all x satisfying the above conditions

$$\mathbf{E}_{p(x)\ell}\, \Gamma(x;\Pi) = \text{Const}\,.$$

References

- D.E. McClure and R.A. Vitale. Polygonal approximation of plane convex bodies. *J. Math. Anal. Appl.*, **51**, 326–358 (1975)
- I. Molchanov and S. Zuyev. Variational analysis of functionals of a Poisson process. *Math. Oper. Research*, **25**, 485–508 (2000)

❏ http://www.stams.strath.ac.uk/~sergei