Measures everywhere

Variation analysis for Poisson processes

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Finite Poisson processes

Poisson process models an array of points scattered randomly and independently with a density proportional to $\mu(dx)$ in a given region X.

More exactly:

Definition: Π is a Poisson process in $[X, \mathcal{B}]$ with the intensity measure μ: $\mu(X) < \infty$ if for any disjoint $B_1, \dots, B_n \in \mathcal{B}$, the number of points of Π in these sets are independent Poisson random variables $\Pi(B_1), \dots, \Pi(B_n)$ with parameters $\mu(B_1), \dots, \mu(B_n)$.

The definition implies:

- If μ is *diffuse*, i. e. $\mu(\{x\}) = 0$ for any singleton $\{x\}$, then with probability 1 no realisation of Π contains multiple points.
- $\mathbf{E}\,\Pi(B) = \mu(B)$, that is why $\mu(dx)$ is also called the mean measure.

❑ We treat each realisation $\{x_1, \dots, x_{\Pi(X)}\}$ of the process Π as a *(random) counting measure* and write $\Pi = \sum_i \delta_{x_i}$, so that $\Pi(B)$ equals the number of points in B and

$$\int f(x)\,\Pi(dx) = \sum_{x_i \in \Pi} f(x_i)\,.$$

Palm distribution

Given an event Ξ, for every $B \in \mathcal{B}$ one can define *Campbell measure* $\mathcal{C}(\Xi, B) = \mathbf{E}_\mu \, 1\!\!1_\Xi(\Pi)\Pi(B)$. This is a measure on $\mathcal{B}$ and $\mathcal{C}(\Xi, \bullet) \ll \mu(\bullet)$, therefore there exists a Radon-Nikodym derivative

$$\frac{d\mathcal{C}(\Xi, \bullet)}{d\mu}(x) = \mathbf{P}_\mu^x(\Xi)$$

which can be chosen to be a *probability* distribution on events Ξ. $\mathbf{P}_\mu^x$ is called the *Palm distribution* of Π and has a meaning of the conditional distribution of Π ‘given there is a point of the process in x’.

Another interpretation is that of the distribution of a configuration seen from a typical point of the process.

Campbell formula

From definition

$$\mathbf{E} \int_B \mathbb{1}_\Xi(\Pi)\, \Pi(dx) = \int_B \mathcal{C}(\Xi, dx) = \int_B \mathbf{P}_\mu^x(\Xi)\, \mu(dx)$$

and thus by the standard monotone class argument

$$\mathbf{E}_\mu \int f(x, \Pi)\, \Pi(dx) = \int \mathbf{E}_\mu^x\, f(x, \Pi)\, \mu(dx) \qquad (1)$$

which is known as *Refined Cambell formula* – continuous analog of the full probability formula. In particular, we have *Campbell formula*:

$$\mathbf{E}_\mu \sum_{x_i \in \Pi} f(x_i) = \mathbf{E}_\mu \int f(x)\, \Pi(dx) = \int f(x)\, \mu(dx)\,.$$

Slivnyak's theorem and Mecke's characterisation

As the points in Poisson process are independent, we should have that the distribution of $\Pi - \delta_x$ under $\mathbf{P}_\mu^x$ should be just $\mathbf{P}_\mu$. This is known as Slivnyak's theorem and equivalent to the following form of Campbell formula (1): for any process $f(x, \Pi)$, one has

$$\mathbf{E}_\mu \int f(x, \Pi)\, \Pi(dx) = \mathbf{E}_\mu \int f(x, \Pi + \delta_x)\, \mu(dx)\,. \qquad (2)$$

Mecke established that (2), in fact, *caracterises* a Poisson process.

Expectation

Given a functional $F = F(\Pi)$, by the full probability formula

$$\mathbf{E}_\mu F = \sum_{n=0}^{\infty} \frac{(\mu(X))^n}{n!} e^{-\mu(X)} \int_{X^n} F\Big(\sum_{i=1}^{n} \delta_{x_i}\Big) \frac{\mu(dx_1)}{\mu(X)} \cdots \frac{\mu(dx_n)}{\mu(X)}$$

$$= e^{-\mu(X)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{X^n} F\Big(\sum_{i=1}^{n} \delta_{x_i}\Big) \mu(dx_1) \ldots \mu(dx_n)^{\text{a}} . \qquad (3)$$

- We view $\mathbf{E}_\mu F$ as a function $\psi(\mu)$ of the intensity measure.

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[a] By definition we have $F(\emptyset)$ for $n = 0$ in the sum above.

Variation analysis

Substituting $\mu \leftarrow (\mu + \eta)$ into (3) and assuming, for simplicity, that F is bounded we get

$$\mathbf{E}_{\mu+\eta} F = e^{-\mu(X)}(1 - \eta(X) + o(\|\eta\|))\times$$

$$\Big[F(\emptyset) + \sum_{n=1} \frac{1}{n!} \int_{X^n} F(\sum_{i=1}^{n} \delta_{x_i})\,(\mu+\eta)(dx_1)\ldots(\mu+\eta)(dx_n)\Big]$$

$$= \mathbf{E}_\mu F + e^{-\mu(X)} \sum_{n=1} \frac{n}{n!} \int_{X^n} F(\sum_{i=1}^{n} \delta_{x_i})\,\mu(dx_1)\ldots\mu(dx_{n-1})\eta(dx_n)$$

$$-\eta(X)e^{-\mu(X)} \sum_{n=0} \frac{1}{n!} \int_{X^n} F(\sum_{i=1}^{n} \delta_{x_i})\,\mu(dx_1)\ldots\mu(dx_n) \quad + o(\|\eta\|)$$

Thus

$$\mathbf{E}_{\mu+\eta}\,F - \mathbf{E}_\mu\,F$$

$$= e^{-\mu(X)} \sum_{n=0} \frac{1}{n!} \int_{X^{n+1}} F\Big(\sum_{i=1}^{n} \delta_{x_i} + \delta_x\Big)\,\mu(dx_1)\ldots\mu(dx_n)\eta(dx)$$

$$-e^{-\mu(X)} \sum_{n=0} \frac{1}{n!} \int_{X^{n+1}} F\Big(\sum_{i=1}^{n} \delta_{x_i}\Big)\,\mu(dx_1)\ldots\mu(dx_n)\eta(dx) \quad +o(\|\eta\|)$$

$$= \mathbf{E}_\mu \int_X [F(\Pi + \delta_x) - F(\Pi)]\,\eta(dx) \quad + o(\|\eta\|)$$

We see that $\mathbf{E}_\mu\,F$ is differentiable and possesses a gradient function

$$\overline{\Delta}_\mu\,(x) = \mathbf{E}_\mu[F(\Pi + \delta_x) - F(\Pi)]$$

which we call the expected first difference.

Analyticity of the expectation

Theorem 1. *Assume that there exist a constant* $b > 0$ *such that* $|F\big(\sum_{i=1}^n \delta_{x_i}\big)| \le b^n$ *for all* $n \ge 0$ *and* $(x_1, \ldots, x_n) \in X^n$*. Then* $\psi(\mu) = \mathbf{E}_\mu F(\Pi)$ *is analytic on* $\mathbb{M}_+$ *and*

$$\mathbf{E}_{\mu+\eta} F = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{X^n} \overline{\Delta_\mu^n}(x_1, \ldots, x_n)\, \eta^n(dx_1 \ldots dx_n)\,, \qquad (4)$$

where

$$\overline{\Delta_\mu^n}(x_1, \ldots, x_n) = \mathbf{E}_\mu\, \Delta_\mu^n(x_1, \ldots, x_n; \Pi)$$
$$= \mathbf{E}_\mu \left[\sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} F\Big(\Pi + \sum_{j=1}^{m} \delta_{x_j}\Big) \right]. \qquad (5)$$

First Fréchet derivatives

In particular,

$$\overline{\Delta}_\mu(x) = \mathbf{E}_\mu[F(\Pi + \delta_x) - F(\Pi)] \quad \text{gradient function}$$

$$\overline{\Delta_\mu^2}(x_1, x_2) = \mathbf{E}_\mu\left[F(\Pi + \delta_{x_1} + \delta_{x_2}) - 2F(\Pi + \delta_{x_1}) + F(\Pi)\right]$$

etc.

- We call $\overline{\Delta_\mu^n}(x_1, \ldots, x_n)$ the *expected nth order difference*.

Perturbation analysis

Consider the case a homogeneous Poisson process with intensity λ in a compact $X \subset \mathbb{R}^d$ so that the intensity measure is $\lambda\ell$ (ℓ is the Lebesgue measure). Slightly abusing notation, write simply $\mathbf{E}_\lambda$ instead of $\mathbf{E}_{\lambda\ell}$. Then

$$\frac{d}{d\lambda}\mathbf{E}_\lambda\, F = \lim_{t\downarrow 0}\frac{1}{t}\Big[E_{\lambda+t}F - \mathbf{E}_\lambda\, F\Big]$$

$$= \lim_{t\downarrow 0}\frac{1}{t}\Big[D\,\mathbf{E}_\lambda\, F[t\ell] + o(t)\Big] = \int_X \mathbf{E}_\lambda[F(\Pi+\delta_x) - F(\Pi)]\,dx$$

Russo's formula for Poisson processes

Let $F(\Pi) = 1\!\!1_\Xi(\Pi)$ for some event Ξ and let $\Upsilon(\Pi) = \{x \in X : \ 1\!\!1_\Xi(\Pi + \delta_x) \neq 1\!\!1_\Xi(\Pi)\}$. Then

$$\frac{d}{d\lambda}\mathbf{P}_\lambda(\Xi) = \int_X \mathbf{E}_\lambda[1\!\!1_\Xi(\Pi + \delta_x) - 1\!\!1_\Xi(\Pi)]\,dx$$

$$= \mathbf{E}_\lambda \int_X [1\!\!1_\Xi(\Pi + \delta_x) - 1\!\!1_\Xi(\Pi)]\, 1\!\!1_{\Upsilon(\Pi)}(x)\,dx$$

$$= \mathbf{E}_\lambda \int_X 1\!\!1_\Xi(\Pi + \delta_x)\, 1\!\!1_{\Upsilon(\Pi)}(x)\,dx - \mathbf{E}_\lambda \int_X 1\!\!1_\Xi(\Pi)\, 1\!\!1_{\Upsilon(\Pi)}(x)\,dx$$

By Slivnyak's theorem (2)

$$\mathbf{E}_\lambda \int_X \mathbb{1}_\Xi(\Pi + \delta_x)\, \mathbb{1}_{\Upsilon(\Pi)}(x)\, dx$$
$$= \frac{1}{\lambda}\, \mathbf{E}_\lambda \int_X \mathbb{1}_\Xi(\Pi)\, \mathbb{1}_{\Upsilon(\Pi-\delta_x)}(x)\, \Pi(dx)$$
$$= \frac{1}{\lambda}\, \mathbf{E}_\lambda\, \mathbb{1}_\Xi(\Pi) N_\Xi(\Pi)\,,$$

where $N_\Xi(\Pi) = \mathrm{card}\{x_i \in \Pi :\ \mathbb{1}_\Xi(\Pi) \neq \mathbb{1}_\Xi(\Pi - \delta_{x_i})\}$ is the number of *pivotal* points for event Ξ in configuration Π, i. e. the points which removal would break the occurence of Ξ.

$$\mathbf{E}_\lambda \int_X \mathbb{1}_\Xi(\Pi)\, \mathbb{1}_{\Upsilon(\Pi)}(x)\, dx = \mathbf{E}_\lambda\, \mathbb{1}_\Xi(\Pi) V_\Xi(\Pi)\,,$$

where $V_\Xi(\Pi) = \mathrm{vol}\{x \in X :\ \mathbb{1}_\Xi(\Pi + \delta_x) \neq \mathbb{1}_\Xi(\Pi)\}$ is the volume of the pivotal locations, where adding a point would break the occurence of Ξ.

Finally,

$$\frac{d}{d\lambda}\mathbf{P}_\lambda(\Xi) = \mathbf{E}_\lambda\, \mathbb{1}_\Xi(\Pi)[\lambda^{-1} N_\Xi(\Pi) - V_\Xi(\Pi)] \tag{6}$$

$$\frac{d}{d\lambda}\log \mathbf{P}_\lambda(\Xi) = \mathbf{E}_\lambda\left[\lambda^{-1} N_\Xi - V_\Xi \,\middle|\, \Xi\right] \tag{7}$$

$$\mathbf{P}_{\lambda_2}(\Xi) = \mathbf{P}_{\lambda_1}(\Xi) \exp\left\{\int_{\lambda_1}^{\lambda_2} \mathbf{E}_\lambda\left[\lambda^{-1} N_\Xi - V_\Xi \,\middle|\, \Xi\right] d\lambda\right\}. \tag{8}$$

Toy example

Consider a set B of volume V and let $\Xi = \{\Pi(B) = k\}$. Surely,

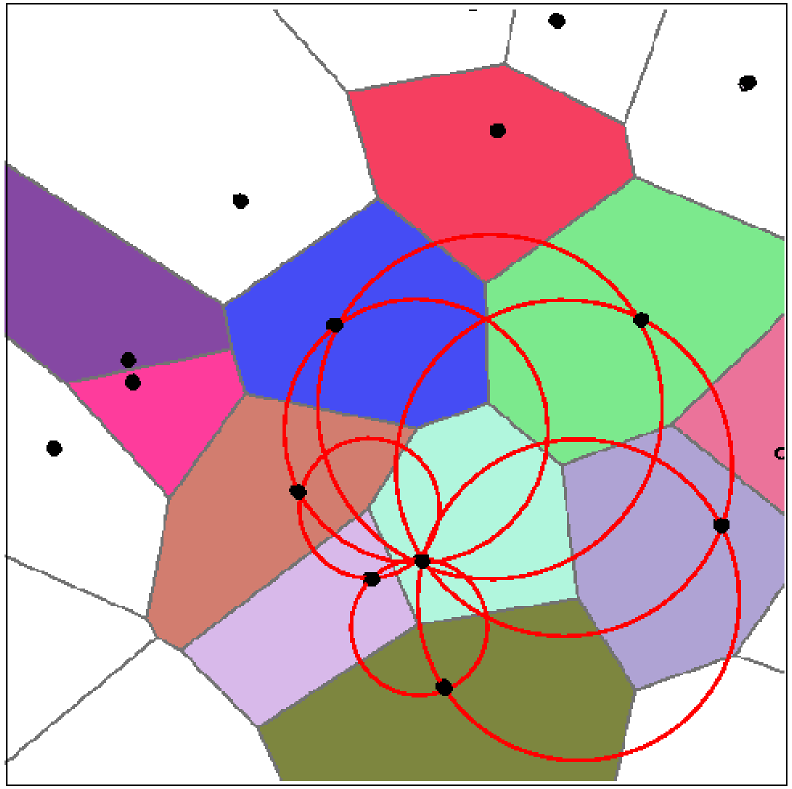
$$\mathbf{P}_\lambda(\Xi) = \frac{(\lambda V)^k}{k!} \exp\{-\lambda V\}.$$

Thus

$$\frac{d}{d\lambda} \log \mathbf{P}_\lambda(\Xi) = \frac{k}{\lambda} - V\,. \qquad (9)$$

On the other hand, when Ξ occurs, there are k points in B and removing any of them would break occurence of Ξ. Thus on Ξ, $N_\Xi = \mathbf{E}_\lambda[N_\Xi \mid \Xi] = k$. Similarly, no additional point could be added anywhere in B without breaking the occurence of Ξ. So on Ξ, $V_\Xi = \mathbf{E}_\lambda[V_\Xi \mid \Xi] = V$ and (7) is seen to be equivalent to (9).

Voronoi flower



By similar method one may derive that the conditional distribution of the volume of a typical Voronoi flower (the one at the origin under $\mathbf{P}^0$) given the corresponding Voronoi cell has n sides is $\mathrm{Gamma}(n, \lambda)$.

Set indexed filtration

Consider a continuous time process $\xi_t,\ t \geq 0$ and filtration $\mathcal{F}_{[0,t]} = \sigma\{\xi_s,\ 0 \leq s \leq t\}$. τ is a stopping time if $\{\tau \leq t\} \in \mathcal{F}_{[0,t]}\ \forall t$ or equivalently, random set $[0,\tau]$ is such that $\{[0,\tau] \subseteq [0,t]\} \in \mathcal{F}_{[0,t]}\ \forall t$.

Consider now a homogeneous Poisson process Π in $\mathbb{R}^d$ and let $\mathcal{F}_B = \sigma\{\Pi(A),\ A \subseteq B\}$ be the natural filtration. We have

- monotonicity: $\mathcal{F}_{K_1} \subseteq \mathcal{F}_{K_2}$ for any two compact $K_1 \subseteq K_2$;
- continuity from above: $\mathcal{F}_K = \cap_{n=1}^{\infty} \mathcal{F}_{K_n}$ if $K_n \downarrow K$.

Stopping sets

Definition: A random compact set Δ is called a *stopping set* (more precisely, $\{\mathcal{F}_K\}$-stopping set) if the event $\{\Delta \subseteq K\}$ is $\mathcal{F}_K$ measurable for all compact K.

Let $\mathcal{F} = \vee_{K \in \mathbb{K}} \mathcal{F}_K$, where $\mathbb{K}$ is the collections of compact sets. The *stopping σ-algebra* is the following collection:

$$\mathcal{F}_\Delta = \{A \in \mathcal{F} : \; A \cap \{\Delta \subseteq K\} \in \mathcal{F}_K \; \forall K \in \mathbb{K}\}.$$

Set-indexed martingales

Definition. A set indexed random process $X_K,\ K \in \mathbb{K}$ is called a *martingale* (more precisely, a $\big(\mathbf{P}, \{\mathcal{F}_K\}\big)$*-martingale*) if for all $K_1,\ K_2 \in \mathbb{K}$ such that $K_1 \subseteq K_2$ one has

$$\mathbf{E}[X_{K_2} \,\big|\, \mathcal{F}_{K_1}] = X_{K_1} \quad \mathbf{P}{-}a.\,s.$$

Theorem 2. *Let $\Delta_1,\ \Delta_2$ be two a. s. compact stopping sets such that $\Delta_1 \subseteq \Delta_2$ almost surely. Let X_K be a uniformly integrable martingale (we omit details here!). Then*

$$\mathbf{E}\left[X_{\Delta_2} \,\big|\, \mathcal{F}_{\Delta_1}\right] = X_{\Delta_1}\ a.\,s. \tag{10}$$

provided $\mathbf{E}\,|X_{\Delta_2}| < \infty$.

Likelihood ratio

An important example of a uniformly integrable martingale is provided by a *likelihood ratio*. Namely, let $\mathbf{Q}$ and $\mathbf{P}$ be two probability measures on $\mathcal{F}$ such that $\mathbf{Q} \underset{loc}{\ll} \mathbf{P}$, i. e. for any $K \in \mathbb{K}$ the restriction $\mathbf{Q}^K$ of $\mathbf{Q}$ onto $\mathcal{F}_K$ is absolutely continuous with respect to the restriction $\mathbf{P}^K$ of $\mathbf{P}$ onto the same σ-algebra. Denote the likelihood ratio by

$$L_K = \frac{d\mathbf{Q}^K}{d\mathbf{P}^K}\,,$$

For Poisson processes we have that

$$L_K = \frac{d\mathbf{P}_\lambda^K}{d\mathbf{P}_\rho^K}(\Pi) = \left(\frac{\lambda}{\rho}\right)^{\Pi(K)} e^{-(\lambda-\rho)\ell(K)},\ \forall K \in \mathbb{K}\,. \qquad (11)$$

Gamma-type result

Theorem 3. *Let* Δ *be an a. s. compact stopping set with respect to the natural filtration of a homogeneous Poisson process* Π *with density* λ *in* $\mathbb{R}^d$*. Assume that*

$$\mathbf{P}_\lambda\{\Pi(\Delta) = n\} > 0 \text{ and does not depend on } \lambda. \qquad (12)$$

Then $\ell(\Delta)$ *given* $\Pi(\Delta) = n$ *has* $\mathrm{Gamma}(n, \lambda)$ *distribution.*

Remark. Condition (12) is satisfied if $\Delta(\Pi)$ is equivariant under scaling: $\Delta(t\Pi) = t\Delta(\Pi)$ for all Π and $t > 0$.

Examples

- The minimal closed ball centred in the origin and containing exactly n Poisson process points is a stopping set and its volume conforms to $\mathrm{Gamma}(n, \lambda)$ distribution (this is trivial).

- A typical Voronoi flower is a stopping set and its volume given that the corresponding Voronoi cell has n sides, is $\mathrm{Gamma}(n, \lambda)$-distributed.

Proof of the Gamma-type result

Kurtz-Doob theorem 2 implies

$$\mathbf{E}_\lambda F = \mathbf{E}_\rho \left(\frac{\lambda}{\rho}\right)^{\Pi(\Delta)} e^{-(\lambda-\rho)\ell(\Delta)} F \qquad (13)$$

for any $\mathcal{F}_\Delta$-measurable F. By (13) for any z we can write

$$\mathbf{E}_\lambda \left[e^{z\ell(\Delta)} \,\middle|\, \Pi(\Delta) = n\right] = \frac{\mathbf{E}_\lambda \left[e^{z\ell(\Delta)}\, \mathbb{1}\{\Pi(\Delta) = n\}\right]}{\mathbf{P}_\lambda\{\Pi(\Delta) = n\}}$$

$$= \frac{\mathbf{E}_\rho \left[e^{z\ell(\Delta)}\, \mathbb{1}\{\Pi(\Delta) = n\}\lambda^n \rho^{-n} e^{-(\lambda-\rho)\ell(\Delta)}\right]}{\mathbf{P}_\rho\{\Pi(\Delta) = n\}}\,.$$

Choosing now $\rho = \lambda - z$ we see that the last expression simplifies to $(1 - z/\lambda)^{-n}$ which is the Laplace transform for $\mathrm{Gamma}(n, \lambda)$.

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