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Partial Differential Equations with Numerical Methods

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1 Introduction

In this first chapter we begin in Sect. 1.1 by introducing the partial differential equations and associated initial and boundary value problems that we shall study in the following chapters. The equations are classified into elliptic, parabolic, and hyperbolic equations, and we indicate the corresponding type of problems in physics that they model. We discuss briefly the concept of a well posed boundary value problem, and the various techniques used in our subsequent presentation. In Sect. 1.2 we introduce some notation and concepts that will be used throughout the text, and in Sect. 1.3 we include a detailed derivation of the heat equation from physical principles explaining the meaning of all terms that occur in the equation and the boundary conditions. In the problem section, Sect. 1.4, we add some further illustrative material.

1.1 Background

In this text we study boundary value and initial-boundary value problems for partial differential equations, that are significant in applications, from both a theoretical and a numerical point of view. As a typical example of such a boundary value problem we consider first Dirichlet's problem for Poisson's equation,

$$-\Delta u = f(x) \quad \text{in } \Omega, \tag{1.1}$$

$$u = g(x) \quad \text{on } \Gamma, \tag{1.2}$$

where $x = (x_1, \dots, x_d)$, Δ is the Laplacian defined by $\Delta u = \sum_{j=1}^{d} \partial^2 u / \partial x_j^2$, and Ω is a bounded domain in d-dimensional Euclidean space $\mathbf{R}^d$ with boundary Γ. The given functions $f = f(x)$ and $g = g(x)$ are the *data* of the problem. Instead of Dirichlet's boundary condition (1.2) one can consider, for instance, Neumann's boundary condition

$$\frac{\partial u}{\partial n} = g(x) \quad \text{on } \Gamma, \tag{1.3}$$

where $\partial u / \partial n$ denotes the derivative in the direction of the exterior unit normal n to Γ. Another choice is Robin's boundary condition

$$\frac{\partial u}{\partial n} + \beta(x) u = g(x) \quad \text{on } \Gamma. \tag{1.4}$$

More generally, a linear second order elliptic equation is of the form

$$\mathcal{A}u := -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) + \sum_{j=1}^{d} b_j(x)\frac{\partial u}{\partial x_j} + c(x)u = f(x), \tag{1.5}$$

where $A(x) = (a_{ij}(x))$ is a sufficiently smooth positive definite matrix, and such an equation may also be considered in Ω together with various boundary conditions. In our treatment below we shall often restrict ourselves, for simplicity, to the isotropic case $A(x) = a(x)I$, where $a(x)$ is a smooth positive function and I the identity matrix.

Elliptic equations such as the above occur in a variety of applications, modeling, for instance, various potential fields (gravitational, electrostatic, magnetostatic, etc.), probability densities in random-walk problems, stationary heat flow, and biological phenomena. They are also related to important areas within pure mathematics, such as the theory of functions of a complex variable $z = x + iy$, conformal mapping, etc. In applications they often describe stationary, or time independent, physical states.

We also consider time dependent problems, and our two model equations are the heat equation,

$$\frac{\partial u}{\partial t} - \Delta u = f(x,t), \tag{1.6}$$

and the wave equation,

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = f(x,t). \tag{1.7}$$

These will be considered for positive time t, and for x varying either throughout $\mathbf{R}^d$ or in some bounded domain $\Omega \subset \mathbf{R}^d$, on the boundary of which boundary conditions are prescribed as for Poisson's equation above. For these time dependent problems, the value of the solution u has to be given at the initial time $t = 0$, and in the case of the wave equation, also the value of $\partial u/\partial t$ at $t = 0$. In the case of the unrestricted space $\mathbf{R}^d$ the respective problems are referred to as the pure *initial value problem* or *Cauchy problem* and, in the case of a bounded domain Ω, a mixed *initial-boundary value problem.*

Again, these equations, and their generalizations permitting more general elliptic operators than the Laplacian Δ, appear in a variety of applied contexts, such as, in the case of the heat equation, in the conduction of heat in solids, in mass transport by diffusion, in diffusion of vortices in viscous fluid flow, in telegraphic transmission in cables, in the theory of electromagnetic waves, in hydromagnetics, in stochastic and biological processes; and, in the case of the wave equation, in vibration problems in solids, in sound waves in a tube, in the transmission of electricity along an insulated, low resistance cable, in long water waves in a straight canal, etc.

Some characteristics of equations of type (1.7) are shared with certain systems of first order partial differential equations. We shall therefore also have reason to study scalar linear partial differential equations of the form

$$\frac{\partial u}{\partial t} + \sum_{j=1}^{d} a_j(x,t)\frac{\partial u}{\partial x_j} + a_0(x,t)u = f(x,t),$$

and corresponding systems where the coefficients are matrices. Such systems appear, for instance, in fluid dynamics and electromagnetic field theory.

Applied problems often lead to partial differential equations which are nonlinear. The treatment of such equations is beyond the scope of this presentation. In many cases, however, it is useful to study linearized versions of these, and the theory of linear equations is therefore relevant also to nonlinear problems.

In applications, the equations used in the models normally contain physical parameters. For instance, in the case of the heat conduction problem, the temperature at a point of a homogeneous isotropic solid, extended over Ω, with the thermal conductivity k, density ρ, and specific heat capacity c, and with a heat source $f(x,t)$, satisfies

$$\rho\, c\,\frac{\partial u}{\partial t} = \nabla\cdot(k\nabla u) + f(x,t) \quad \text{in } \Omega.$$

If ρ, c, and k are constant, this equation may be written in the form (1.6) after a simple transformation, but if they vary with x, a more general elliptic operator is involved.

In Sect. 1.3 below we derive the heat equation from physical principles and explain, in the context given, the physical meaning of all terms in the elliptic operator (1.5) as well as the boundary conditions (1.2), (1.3), and (1.4). A corresponding derivation or the wave equation is given in Problem 1.2. Boundary value problems for elliptic equations, or stationary problems, may appear as limiting cases of the evolution problems as $t \to \infty$.

One characteristic of mathematical modeling is that once the model is established, in our case as an initial or initial-boundary value problem for a partial differential equation, the analysis becomes purely mathematical and is independent of any specific application that the model describes. The results obtained are then valid for all the different examples of the model. We shall therefore not use much terminology from physics or other applied fields in our exposition, but invoke special applications in the exercises. It is often convenient to keep such examples in mind to enhance the intuitive understanding of a mathematical model.

The equations (1.1), (1.6), and (1.7) are said to be of elliptic, parabolic, and hyperbolic type, respectively. We shall return to the classification of partial differential equations into different types in Chapt. 11 below, and note here only that a differential equation in two variables x and t of the form

$$a\frac{\partial^2 u}{\partial t^2} + 2b\frac{\partial^2 u}{\partial x \partial t} + c\frac{\partial^2 u}{\partial x^2} + \ldots = f(x,t)$$

is said to be *elliptic, hyperbolic* or *parabolic* depending on whether $\delta = ac - b^2$ is positive, negative, or zero. Here $\ldots$ stands for a linear combination of derivatives of orders at most 1. In particular,

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = f(x,t),$$

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = f(x,t),$$

and

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x,t),$$

are of these three types, respectively. Note that the conditions on the sign of δ are the same as those occurring in the classification of plane quadratic curves into ellipses, hyperbolas, and parabolas.

Together with the partial differential equations we also study numerical approximations by finite difference and finite element methods. For these problems, the continuous and the discretized equations, we prove results of the following types:

- *existence* of solutions,
- *uniqueness* of solutions,
- *stability*, or continuous dependence of solutions with respect to perturbations of data,
- *error estimates* (for numerical methods).

A boundary value problem that satisfies the three first of these conditions is said to be *well posed.* In order to prove such results we employ several techniques:

- *maximum principles*,
- *Fourier methods*; these are techniques that are based on the use of the Fourier transform, Fourier series expansion, or eigenfunction expansion,
- *energy estimates*,
- representation of solution operators by means of *Green's functions*.

1.2 Notation and Mathematical Preliminaries

In this section we briefly introduce some basic notation that will be used throughout the book. For more details on function spaces and norms we refer to App. A.

By $\mathbf{R}$ and $\mathbf{C}$ we denote the sets of real and complex numbers, respectively, and we write

$$\mathbf{R}^d = \{x = (x_1, \dots, x_d) : x_i \in \mathbf{R},\ i = 1, \dots, d\}, \quad \mathbf{R}_+ = \{t \in \mathbf{R} : t > 0\}.$$

A subset of $\mathbf{R}^d$ is called a domain if it is open and connected. By Ω we usually denote a bounded domain in $\mathbf{R}^d$, for $i = 1, 2$, or 3 (if $d = 1$, then Ω is a bounded open interval). Its boundary $\partial\Omega$ is usually denoted Γ. We assume throughout that Γ is either smooth or a polygon (if $d = 2$) or polyhedron (if $d = 3$). By $\bar{\Omega}$ we denote the closure of Ω, i.e., $\bar{\Omega} = \Omega \cup \Gamma$. The (length, area, or) volume of Ω is denoted by $|\Omega|$, the volume element in $\mathbf{R}^d$ is $\mathrm{d}x = \mathrm{d}x_1 \cdots \mathrm{d}x_d$, and $\mathrm{d}s$ denotes the element of arclength (if $d = 2$) or surface area (if $d = 3$) on Γ. For vectors in $\mathbf{R}^d$ we use the Euclidean inner product $x \cdot y = \sum_{i=1}^d x_i y_i$ and norm $|x| = \sqrt{x \cdot x}$.

Let u, v be scalar functions and $w = (w_1, \dots, w_d)$ a vector-valued function of $x \in \mathbf{R}^d$. We define the gradient, the divergence, and the Laplace operator (Laplacian) by

$$\nabla v = \operatorname{grad} v = \Big(\frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_d}\Big),$$
$$\nabla \cdot w = \operatorname{div} w = \sum_{i=1}^d \frac{\partial w_i}{\partial x_i},$$
$$\Delta v = \nabla \cdot \nabla v = \sum_{i=1}^d \frac{\partial^2 v}{\partial x_i^2}.$$

We recall the *divergence theorem*

$$\int_\Omega \nabla \cdot w \,\mathrm{d}x = \int_\Gamma w \cdot n \,\mathrm{d}s,$$

where $n = (n_1, \dots, n_d)$ is the outward unit normal to Γ. Applying this to the product wv we obtain *Green's formula*:

$$\int_\Omega w \cdot \nabla v \,\mathrm{d}x = \int_\Gamma w \cdot n\, v \,\mathrm{d}s - \int_\Omega \nabla \cdot w\, v \,\mathrm{d}x.$$

When applied with $w = \nabla u$ the formula becomes

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Gamma \frac{\partial u}{\partial n} v \,\mathrm{d}s - \int_\Omega \Delta u\, v \,\mathrm{d}x,$$

where $\partial u/\partial n = n \cdot \nabla u$ is the exterior normal derivative of u on Γ.

A *multi-index* $\alpha = (\alpha_1, \dots, \alpha_d)$ is a d-vector where the α_i are non-negative integers. The *length* $|\alpha|$ of a multi-index α is defined by $|\alpha| = \sum_{i=1}^d \alpha_i$. Given a function $v : \mathbf{R}^d \to \mathbf{R}$ we may write its partial derivatives of order $|\alpha|$ as

$$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}. \tag{1.8}$$

A linear partial differential equation of order k in Ω can therefore be written

$$\sum_{|\alpha| \le k} a_\alpha(x) D^\alpha u = f(x),$$

where the coefficients $a_\alpha(x)$ are functions of x in Ω. We also use subscripts to denote partial derivatives, e.g.,

$$v_t = D_t v = \frac{\partial v}{\partial t}, \quad v_{xx} = D_x^2 v = \frac{\partial^2 v}{\partial x^2}.$$

For $M \subset \mathbf{R}^d$ we denote by $\mathcal{C}(M)$ the linear space of continuous functions on M, and for bounded continuous functions we define the maximum-norm

$$\|v\|_{\mathcal{C}(M)} = \sup_{x \in M} |v(x)|. \tag{1.9}$$

For example, this defines $\|v\|_{\mathcal{C}(\mathbf{R}^d)}$. When M is a bounded and closed set, i.e., a compact set, the supremum in (1.9) is attained and we may write

$$\|v\|_{\mathcal{C}(M)} = \max_{x \in M} |v(x)|.$$

For a not necessarily bounded domain Ω and k a non-negative integer we denote by $\mathcal{C}^k(\Omega)$ the set of k times continuously differentiable functions in Ω. For a bounded domain Ω we write $\mathcal{C}^k(\bar{\Omega})$ for the functions $v \in \mathcal{C}^k(\Omega)$ such that $D^\alpha v \in \mathcal{C}(\bar{\Omega})$ for all $|\alpha| \le k$. For functions in $\mathcal{C}^k(\bar{\Omega})$ we use the norm

$$\|v\|_{\mathcal{C}^k(\bar{\Omega})} = \max_{|\alpha| \le k} \|D^\alpha v\|_{\mathcal{C}(\bar{\Omega})},$$

and the seminorm, including only the derivatives of highest order,

$$|v|_{\mathcal{C}^k(\bar{\Omega})} = \max_{|\alpha| = k} \|D^\alpha v\|_{\mathcal{C}(\bar{\Omega})}.$$

When we are working on a fixed domain Ω we often omit the set in the notation and write simply $\|v\|_{\mathcal{C}}$, $|v|_{\mathcal{C}^k}$, etc.

By $\mathcal{C}_0^k(\Omega)$ we denote the set of functions $v \in \mathcal{C}^k(\Omega)$ that vanish outside some compact subset of Ω, in particular, such functions satisfy $D^\alpha v = 0$ on the boundary of Ω for $|\alpha| \le k$. Similarly, $\mathcal{C}_0^\infty(\mathbf{R}^d)$ is the set of functions that have continuous derivatives of all orders and vanish outside some bounded set.

We say that a function is *smooth* if, depending on the situation, it has sufficiently many continuous derivatives.

We also frequently employ the space $L_2(\Omega)$ of square integrable functions with scalar product and norm

$$(v,w) = (v,w)_{L_2(\Omega)} = \int_\Omega vw \,\mathrm{d}x, \quad \|v\| = \|v\|_{L_2(\Omega)} = \Big(\int_\Omega v^2 \,\mathrm{d}x\Big)^{1/2}.$$

For Ω a domain we also employ the Sobolev space $H^k(\Omega)$, $k \geq 1$, of functions v such that $D^\alpha v \in L_2(\Omega)$ for all $|\alpha| \leq k$, equipped with the norm and seminorm

$$\|v\|_k = \|v\|_{H^k(\Omega)} = \Big(\sum_{|\alpha|\leq k} \|D^\alpha v\|^2\Big)^{1/2},$$

$$|v|_k = |v|_{H^k(\Omega)} = \Big(\sum_{|\alpha|= k} \|D^\alpha v\|^2\Big)^{1/2}.$$

Additional norms are defined and used locally when the need arises.

We use the letters c, C to denote various positive constants that need not be the same at each occurrence.

1.3 Physical Derivation of the Heat Equation

Many equations in physics are derived by combining a conservation law with constitutive relations. A conservation law states that a physical quantity, such as energy, mass, or momentum, is conserved as the physical process develops in time. Constitutive relations express our assumptions about how the material behaves when the state variables change.

In this section we we consider the conduction of heat in a body $\Omega \subset \mathbf{R}^3$ with boundary Γ and derive the heat equation using conservation of energy together with linear constitutive relations.

Conservation of Energy

Consider the balance of heat in an arbitrary subset $\Omega_0 \subset \Omega$ with boundary Γ_0. The energy principle says that the rate of change of the total energy in Ω_0 equals the inflow of heat through Γ_0 plus the heat power produced by heat sources inside Ω_0. To express this in mathematical terms we introduce some physical quantities, each of which is followed, within brackets, by the associated standard unit of measurement.

With $e = e(x,t)$ $[\mathtt{J/m^3}]$ the *density of internal energy* at the point x $[\mathtt{m}]$ and time t $[\mathtt{s}]$, the total amount of heat in Ω_0 is $\int_{\Omega_0} e \,\mathrm{d}x$ $[\mathtt{J}]$. Further with the vector field $j = j(x,t)$ $[\mathtt{J/(m^2s)}]$ denoting the *heat flux* and n the exterior unit normal to Γ_0, the net outflow of heat through Γ_0 is $\int_{\Gamma_0} j \cdot n \,\mathrm{d}s$ $[\mathtt{J/s}]$. Introducing also the power density of heat sources $p = p(x,t)$ $[\mathtt{J/(m^3s)}]$, the energy principle then states that

$$\frac{d}{dt}\int_{\Omega_0} e \,\mathrm{d}x = -\int_{\Gamma_0} j \cdot n \,\mathrm{d}s + \int_{\Omega_0} p \,\mathrm{d}x.$$

Applying the divergence theorem we obtain

$$\int_{\Omega_0} \left(\frac{\partial e}{\partial t} + \nabla \cdot j - p \right) \mathrm{d}x = 0, \quad \text{for } t > 0.$$

Since $\Omega_0 \subset \Omega$ is arbitrary this implies

$$\frac{\partial e}{\partial t} + \nabla \cdot j = p \quad \text{in } \Omega, \quad \text{for } t > 0. \tag{1.10}$$

Constitutive Relations

The internal energy density e depends on the absolute temperature T $[\mathtt{K}]$ and the spatial coordinates, and in our first constitutive relation we assume that e depends linearly on T near a suitably chosen reference temperature T_0, that is,

$$e = e_0 + \sigma(T - T_0) = e_0 + \sigma\,\vartheta, \quad \text{where } \vartheta = T - T_0. \tag{1.11}$$

The coefficient $\sigma = \sigma(x)$ $[\mathtt{J/(m^3\,K)}]$ is called the *specific heat capacity.* (It is usually expressed in the form $\sigma = \rho\, c$, where ρ $[\mathtt{kg/m^3}]$ is mass density and c $[\mathtt{J/(kg\,K)}]$ is the specific heat capacity per unit mass.)

According to *Fourier's law* the heat flux due to conduction is proportional to the temperature gradient, which gives a second constitutive relation,

$$j = -\lambda \nabla \vartheta.$$

The coefficient $\lambda = \lambda(x)$ $[\mathtt{J/(m\,K\,s)}]$ is called the *heat conductivity.* In some situations (e.g., gas in a porous medium, heat transport in a fluid) heat is also transported by convection with heat flux $v\,e$, where $v = v(x,t)$ $[\mathtt{m/s}]$ is the convective velocity vector field. The constitutive relation then reads

$$j = -\lambda \nabla \vartheta + v\,e. \tag{1.12}$$

Substituting (1.11) and (1.12) into (1.10) we obtain

$$\sigma \frac{\partial \vartheta}{\partial t} - \nabla \cdot (\lambda \nabla \vartheta) + \nabla \cdot (\sigma\, v\, \vartheta) = q \quad \text{in } \Omega, \quad \text{where } q = p - \nabla \cdot (v\, e_0), \tag{1.13}$$

which is the *heat equation* with convection.

Boundary Conditions

In the modelling of heat conduction, the differential equation (1.13) is combined with an *initial condition* at time $t = 0$,

$$\vartheta(x, 0) = \vartheta_{\mathrm{i}}(x), \tag{1.14}$$

and a *boundary condition*, expressing that the heat flux through the boundary is proportional to the difference between the surface temperature and the ambient temperature, $j \cdot n = \kappa(\vartheta - \vartheta_{\mathrm{a}})$, where $\kappa = \kappa(x,t)$ $[\mathtt{J/(m^2\,s\,K)}]$ is a heat transfer coefficient. Assuming that the material flow does not penetrate the boundary, i.e., $v \cdot n = 0$, we obtain from (1.12)

$$j \cdot n = -\lambda \nabla\vartheta \cdot n = -\lambda \frac{\partial \vartheta}{\partial n} \quad \text{on } \Gamma,$$

where $\partial\vartheta/\partial n = \nabla\vartheta \cdot n$ denotes the exterior normal derivative of ϑ. Therefore the boundary condition is *Robin's boundary condition*

$$\lambda \frac{\partial \vartheta}{\partial n} + \kappa(\vartheta - \vartheta_{\mathrm{a}}) = 0 \quad \text{on } \Gamma. \tag{1.15}$$

The limit case $\kappa = 0$ means that the boundary surface is perfectly insulated, so that we have *Neumann's boundary condition*,

$$\frac{\partial \vartheta}{\partial n} = 0.$$

At the other extreme, dividing by κ in (1.15) and letting $\kappa \to \infty$, we obtain *Dirichlet's boundary condition*

$$\vartheta = \vartheta_{\mathrm{a}}. \tag{1.16}$$

The limit case $\kappa = \infty$ thus means that the body is in perfect thermal contact with the surroundings, i.e., heat flows freely through the surface, so that the surface temperature of the body is equal to the ambient temperature.

Dimensionless Form

It is often useful to write the above equations in dimensionless form. Choosing reference constants L $[\mathtt{m}]$, τ $[\mathtt{s}]$, ϑ_{f} $[\mathtt{K}]$, σ_{f} $[\mathtt{J/(m^3\,K)}]$, v_{f} $[\mathtt{m/s}]$, etc., we define dimensionless variables

$$\tilde{t} = t/\tau, \quad \tilde{x} = x/L, \quad u(\tilde{x},\tilde{t}) = \vartheta(\tilde{x}L, \tilde{t}\tau)/\vartheta_{\mathrm{f}}.$$

In order to make the heat equation (1.13) dimensionless we divide it by $\lambda_{\mathrm{f}}\vartheta_{\mathrm{f}}/L^2$. Using the chain rule,

$$\frac{\partial u}{\partial \tilde{t}} = \tau \frac{\partial}{\partial t}\Big(\frac{\vartheta}{\vartheta_{\mathrm{f}}}\Big), \quad \tilde{\nabla} u = L\nabla\Big(\frac{\vartheta}{\vartheta_{\mathrm{f}}}\Big),$$

we get

$$d\frac{\partial u}{\partial \tilde{t}} - \tilde{\nabla} \cdot (a\tilde{\nabla} u) + \tilde{\nabla} \cdot (bu) = f \quad \text{in } \tilde{\Omega}, \tag{1.17}$$

where

$$d = \frac{L^2 \sigma_{\mathrm{f}}}{\tau \lambda_{\mathrm{f}}} \frac{\sigma}{\sigma_{\mathrm{f}}}, \quad a = \frac{\lambda}{\lambda_{\mathrm{f}}}, \quad b = \frac{v_{\mathrm{f}} \sigma_{\mathrm{f}} L}{\lambda_{\mathrm{f}}} \frac{\sigma}{\sigma_{\mathrm{f}}} \frac{v}{v_{\mathrm{f}}}, \quad f = \frac{L^2}{\lambda_{\mathrm{f}} \vartheta_{\mathrm{f}}} q.$$

It is natural to choose $\tau = L^2 \sigma_{\mathrm{f}} / \lambda_{\mathrm{f}}$, so that $d = 1$ if $\sigma = \sigma_{\mathrm{f}}$ is constant. The dimensionless number $\mathrm{Pe} = v_{\mathrm{f}} \sigma_{\mathrm{f}} L / \lambda_{\mathrm{f}}$ that appears in the definition of b is called Peclet's number and measures the relative strengths of convection and conduction. Skipping the tilde from now on, we write (1.17) as

$$d \frac{\partial u}{\partial t} - \nabla \cdot (a \nabla u) + b \cdot \nabla u + cu = f \quad \text{in } \Omega, \quad \text{where } c = \nabla \cdot b. \tag{1.18}$$

The boundary condition (1.15) and the initial condition (1.14) transform in a similar way to

$$a \frac{\partial u}{\partial n} + h(u - u_{\mathrm{a}}) = 0 \quad \text{on } \Gamma, \tag{1.19}$$

and

$$u(x, 0) = u_{\mathrm{i}}(x). \tag{1.20}$$

Here $h = \mathrm{Bi}\, \kappa / \kappa_{\mathrm{f}}$, where $\mathrm{Bi} = L \kappa_{\mathrm{f}} / \lambda_{\mathrm{f}}$ is called the Biot number.

The partial differential equation (1.18) together with the initial condition (1.20) and the boundary condition (1.19) is called an *initial-boundary value problem.* The term $-\nabla \cdot (a \nabla u)$ is written in *divergence form.* This form arises naturally in the derivation of the equation, and it is convenient in much of the mathematical analysis, as we shall see below. However, we sometimes expand the derivative and write the equation in non-divergence form:

$$d \frac{\partial u}{\partial t} - a \Delta u + \bar{b} \cdot \nabla u + cu = f, \quad \text{where } \bar{b} = b - \nabla a. \tag{1.21}$$

Some Simplified Problems

It is useful to study various simplifications of the above equations, because it may then be possible to carry the mathematical analysis further than in the general case. If we assume that the coefficients are constant, with $b = 0$, $c = 0$, then (1.18) reduces to (recall that $d = 1$ if σ is constant)

$$\frac{\partial u}{\partial t} - a \Delta u = f. \tag{1.22}$$

For $a = 1$ this is equation (1.6). If f and the boundary condition are independent of t, then u could be expected to approach a stationary state as t grows, i.e., $u(x,t) \to v(x)$ as $t \to \infty$, and since we should then have $\partial u / \partial t \to 0$, we find that v satisfies *Poisson's equation* (1.1). If in addition $f = 0$, we have *Laplace's equation*

$$-\Delta u = 0.$$

Solutions of Laplace's equation are called *harmonic functions.*

Another important kind of simplification is obtained by reduction of dimension. For example, consider stationary (time-independent) heat conduction in a (not necessarily circular) cylinder oriented along the x_1-axis with insulated mantle surface. If the coefficients a, b, c, f in (1.18) are independent of x_2 and x_3, then it is reasonable to assume that the solution u also depends only on one variable x_1, which we then denote by x, i.e., $u = u(x)$. The heat equation (1.18) then reduces to an ordinary differential equation

$$-(au')' + bu' + cu = f \quad \text{in } \Omega = (0,1).$$

The boundary condition (1.19) becomes

$$-a(0)u'(0) + h_0(u(0) - u_0) = 0, \quad a(1)u'(1) + h_1(u(1) - u_1) = 0. \tag{1.23}$$

We call this a *two-point boundary value problem.* Similar simplifications are obtained under cylindrical and spherical symmetry by writing the equations in cylindrical respectively spherical coordinates. If the coefficients are constant, then we can readily express the solution in terms of well-known special functions, see Problem 1.6.

Nonlinear Equations, Linearization

The coefficients in the heat equation (1.18) and in the boundary conditions often depend on the temperature u, which makes the equations nonlinear. Although the study of nonlinear equations is outside the scope of this book, we mention that the study of nonlinear equations often proceeds by *linearization*, i.e., by reduction to the study of related linear equations. We illustrate this in the case of the equation

$$F(u) := \frac{\partial u}{\partial t} - \nabla \cdot \big(a(u)\nabla u\big) - f(u) = 0 \quad \text{in } \Omega, \quad \text{for } t > 0,$$

which is of the form (1.18), and which is to be solved together with suitable initial and boundary conditions. One approach to such a problem is to use Newton's method, which produces a sequence of approximate solutions u^k from a starting guess u^0 in the following way: Given u^k we want to find an increment v^k such that $u^{k+1} = u^k + v^k$ is a better approximation of the exact solution than u^k. Approximating $F(u^{k+1}) = 0$ by $F(u^k) + F'(u^k)v^k = 0$, we obtain a linearized equation

$$\frac{\partial v^k}{\partial t} - \nabla \cdot \big(a(u^k)\nabla v^k\big) - \nabla \cdot \big(a'(u^k)\nabla u^k v^k\big) - f'(u^k)v^k = -F(u^k) \quad \text{in } \Omega,$$

which is solved together with an initial condition and linearized boundary conditions. This equation is a linear equation in v^k of the form (1.18), where the new coefficients $a(u^k(x,t))$, etc., depend on x and t.

1.4 Problems

Problem 1.1. (Derivation of the convection-diffusion equation.) Let $c = c(x,t)$ [mol/m^3] denote the concentration at the point x [m] and time t [s] of a substance that is being transported through a domain $x \in \Omega \subset \mathbf{R}^3$ by convection and diffusion. The flux due to convection is

$$j_c = vc, \quad [\mathtt{mol/(m^2s)}]$$

where $v = v(x)$ [m/s] is the convective velocity field. The flux due to diffusion is (Fick's law)

$$j_d = -D\nabla c, \quad [\mathtt{mol/(m^2s)}]$$

where $D = D(x)$ [m^2/s] is the diffusion coefficient. Let r [mol/(m^{3}s)] denote the rate of creation/annihilation of material, e.g., by chemical reaction. The total mass of the substance within an arbitrary subdomain is $\int_{\Omega_0} c\,\mathrm{d}x$. Use the conservation of mass and the divergence theorem to derive the convection-diffusion equation

$$\frac{\partial c}{\partial t} - \nabla \cdot (D\nabla c) + \nabla \cdot (vc) = r, \quad [\mathtt{mol/(m^3s)}]$$

which is of the same mathematical form as (1.13). Derive a boundary condition of the form (1.15). Show that these equations can be written in the same dimensionless form as (1.18) and (1.19).

Problem 1.2. (Derivation of the wave equation.) Consider the longitudinal motion of an elastic bar of length L [m] and of constant cross-sectional area A [m^2] and with density ρ [kg/m^3]. Let $u = u(x,t)$ [m] denote the displacement at time t [s] of a cross-section originally located at $x \in [0, L]$. Newton's law of motion states that

$$\frac{d}{dt}\int_a^b pA\,\mathrm{d}x = \big(\sigma(b) - \sigma(a)\big)A, \quad [\mathtt{N}]$$

where $\int_a^b pA\,\mathrm{d}x$ [kg m/s] is the total momentum of an arbitrary segment (a,b) and σ [N/m^2] is the stress (force per unit cross-sectional area). This leads to

$$\frac{\partial p}{\partial t} = \frac{\partial \sigma}{\partial x}.$$

For small displacements we have a linear relationship between the stress σ and the strain $\epsilon = \partial u/\partial x$, namely Hooke's law,

$$\sigma = E\epsilon,$$

where E [N/m^2] is the modulus of elasticity, and the momentum density is given by $p = \rho\partial u/\partial t$. Show that u satisfies the wave equation

$$\rho\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\Big(E\frac{\partial u}{\partial x}\Big).$$

Discuss various possible boundary conditions at the ends of the bar. For example, at $x = L$:

– fixed end, $u(L) = 0$,
– free end, $\sigma(L) = 0$, which leads to $u_x(L) = 0$,
– elastic support, $\sigma(L) = -ku(L)$, which leads to $Eu_x(L) + ku(L) = 0$.

Note that these are of the form (1.23).

Problem 1.3. (Elastic beam.) Consider the bending of an elastic beam that extends along the interval $0 \leq x \leq L$. At an arbitrary cross-section at a distance x from the left end we introduce the bending moment (torque) $M = M(x)$ [Nm], the transversal force $T = T(x)$ [N], and the external applied force $q = q(x)$ per unit length [N/m]. It can be shown that equilibrium of forces requires $M' = T$ and $T' = -q$. Let $u = u(x)$ [m] be the small transversal deflection of the beam. The bending angle is then approximately u'. The constitutive law is $M = -EIu''$, where E [N/m^2] the modulus of elasticity and I [m^4] is a moment of inertia of the cross-section of the beam. Show that this leads to the fourth order equation

$$(EIu'')'' = q.$$

Discuss various possible boundary conditions at the ends of the beam. For example, at $x = L$:

– clamped end, $u(L) = 0$, $u'(L) = 0$,
– free end, $M(L) = -(EIu'')(L) = 0$, $T(L) = -(EIu'')'(L) = 0$,
– hinge, $u'(L) = 0$, $M(L) = -(EIu'')(L) = 0$.

Problem 1.4. (The Laplace operator in spherical symmetry.) Introduce spherical coordinates (r, θ, ϕ) defined by $x_1 = r\sin\theta\cos\phi$, $x_2 = r\sin\theta\sin\phi$, $x_3 = r\cos\theta$. Assume that the function u does not depend on θ and ϕ, i.e., $u = u(r)$. Show that

$$\Delta u = \frac{1}{r^2}\frac{d}{dr}\Big(r^2\frac{du}{dr}\Big).$$

Problem 1.5. (The Laplace operator in cylindrical symmetry.) Introduce cylindrical coordinates (ρ, φ, z) defined by $x_1 = \rho\cos\varphi$, $x_2 = \rho\sin\varphi$, $x_3 = z$. Assume that the function u does not depend on φ and z, i.e., $u = u(\rho)$. Show that

$$\Delta u = \frac{1}{\rho}\frac{d}{d\rho}\Big(\rho\frac{du}{d\rho}\Big).$$

Problem 1.6. Let $\Omega = \{x \in \mathbf{R}^3 : |x| < 1\}$. Determine an explicit solution of the boundary value problem

$$-\Delta u + c^2 u = f \quad \text{in } \Omega, \quad \text{with } u = g \quad \text{on } \Gamma,$$

assuming spherical symmetry and that c, f, g are constants. That is, solve

$$-(r^2 u'(r))' + c^2 r^2 u(r) = r^2 f \quad \text{for } r \in (0,1), \quad \text{with } u(1) = g,\ u(0) \text{ finite.}$$

Hint: Set $v(r) = ru(r)$.

2 A Two-Point Boundary Value Problem

For the purpose of preparing for the treatment of boundary value problems for elliptic partial differential equations we consider here a simple two-point boundary value problem for a second order linear ordinary differential equation. In the first section we derive a maximum principle for this problem, and use it to show uniqueness and continuous dependence on data. In the second section we construct a Green's function in a special case and show how this implies the existence of a solution. In the third section we write the problem in variational form, and use this together with simple tools from functional analysis to prove existence, uniqueness, and continuous dependence on data.

2.1 The Maximum Principle

We consider the boundary value problem

$$\begin{aligned} \mathcal{A}u := -\,(au')' + bu' + cu = f \quad \text{in } \Omega = (0,1), \\ u(0) = u_0, \ u(1) = u_1, \end{aligned} \tag{2.1}$$

where the coefficients $a = a(x)$, $b = b(x)$, and $c = c(x)$ are smooth functions with

$$a(x) \geq a_0 > 0, \quad c(x) \geq 0, \quad \text{for } x \in \bar{\Omega} = [0,1], \tag{2.2}$$

and where the function $f = f(x)$ and the numbers u_0, u_1 are given, cf. Sect. 1.3.

In the particular case that $a = 1, \ b = c = 0$, this reduces to

$$-u'' = f \quad \text{in } \Omega, \quad \text{with } u(0) = u_0, \ u(1) = u_1. \tag{2.3}$$

By integrating this differential equation twice we find that a solution must have the form

$$u(x) = -\int_0^x \int_0^y f(s)\,\mathrm{d}s\,\mathrm{d}y + \alpha x + \beta, \tag{2.4}$$

with the constants α, β to be determined. Setting $x = 0$ and $x = 1$ we find

$$\alpha = u_1 - u_0 + \int_0^1 \int_0^y f(s)\,\mathrm{d}s\,\mathrm{d}y, \quad \beta = u_0.$$

Reversing the steps we find that (2.4), with these α, β, is the unique solution of (2.3).

In the special case $f = 0$ the solution of (2.3) is the linear function $u(x) = u_0(1-x) + u_1 x$. In particular, the values of this function lie between those at $x = 0$ and $x = 1$, and its maximum and minimum are thus located at the endpoints of the interval Ω. More generally, we have the following maximum (minimum) principle for (2.1).

Theorem 2.1. *Consider the differential operator $\mathcal{A}$ in* (2.1), *and assume that $u \in \mathcal{C}^2 = \mathcal{C}^2(\bar{\Omega})$ and*

$$\mathcal{A}u \leq 0 \quad \big(\mathcal{A}u \geq 0\big) \quad \text{in } \Omega. \tag{2.5}$$

(i) *If $c = 0$, then*

$$\max_{\bar{\Omega}} u = \max\big\{u(0), u(1)\big\} \quad \Big(\min_{\bar{\Omega}} u = \min\big\{u(0), u(1)\big\}\Big). \tag{2.6}$$

(ii) *If $c \geq 0$ in Ω, then*

$$\max_{\bar{\Omega}} u \leq \max\big\{u(0), u(1), 0\big\} \quad \Big(\min_{\bar{\Omega}} u \geq \min\big\{u(0), u(1), 0\big\}\Big). \tag{2.7}$$

In case (i) we conclude that the maximum of u is attained at the boundary, i.e., at one of the endpoints of the interval Ω. In case (ii) we draw the same conclusion if the maximum is nonnegative. This does not exclude the possibility that the maximum is attained also in the interior of Ω. However, there is also a stronger form of the maximum principle, which in case (i) asserts that if (2.5) holds and u has a maximum at an interior point of Ω (in case (ii) a nonnegative interior maximum), then u is constant in $\bar{\Omega}$. We shall not prove this here, but we refer to Sect. 3.3 below for the corresponding result for harmonic functions. The variants within parentheses, with $\mathcal{A}u \geq 0$, may be described as a minimum principle; it is reduced to the maximum principle by looking at $-u$.

Proof. (i) Assume first, instead of (2.5), that $\mathcal{A}u < 0$ in Ω. If u has a maximum at an interior point $x_0 \in \Omega$, then at this point we have $u'(x_0) = 0$ and $u''(x_0) \leq 0$, so that $\mathcal{A}u(x_0) \geq 0$, which contradicts our assumption. Hence u cannot have an interior maximum point and (2.6) follows.

Assume now that we only know that $\mathcal{A}u \leq 0$ in Ω. Let ϕ be a function such that $\phi \geq 0$ in $\bar{\Omega}$ and $\mathcal{A}\phi < 0$ in Ω. For example, we may use the function $\phi(x) = \mathrm{e}^{\lambda x}$ with λ so large that $\mathcal{A}\phi = (-a\lambda^2 + (b - a')\lambda)\phi < 0$ in $\bar{\Omega}$. Assume now that u attains its maximum at an interior point x_0 but not at $x = 0$ or $x = 1$. Then for $\epsilon > 0$ sufficiently small this is true also for $v = u + \epsilon\phi$. But $\mathcal{A}v = \mathcal{A}u + \epsilon\mathcal{A}\phi < 0$ in $\bar{\Omega}$, which contradicts the first part of the proof.

(ii) If $u \leq 0$ in Ω, then (2.7) holds trivially. Otherwise assume that $\max_{\bar{\Omega}} u = u(x_0) > 0$ and $x_0 \neq 0, 1$. Let (α, β) be the largest subinterval of Ω containing x_0 in which $u > 0$. We now have $\tilde{\mathcal{A}}u := \mathcal{A}u - cu \leq 0$ in (α, β). Part (i), applied with the operator $\tilde{\mathcal{A}}$ in the interval (α, β), therefore implies $u(x_0) = \max\{u(\alpha), u(\beta)\}$. But then α and β could not both be interior points of Ω, for then either $u(\alpha)$ or $u(\beta)$ would be positive, and the interval (α, β) would not be as large as possible with $u > 0$. This implies $u(x_0) = \max\{u(0), u(1)\}$ and hence (2.7). □

As a consequence of this theorem we have the following stability estimate with respect to the maximum-norm, $\|v\|_{\mathcal{C}} = \max_{\bar{\Omega}} |v|$, defined in Sect. 1.2.

Theorem 2.2. *Let $\mathcal{A}$ be as in (2.1) and (2.2). If $u \in \mathcal{C}^2$, then*

$$\|u\|_{\mathcal{C}} \leq \max\big\{|u(0)|, |u(1)|\big\} + C\|\mathcal{A}u\|_{\mathcal{C}}.$$

The constant C depends on the coefficients of $\mathcal{A}$ but not on u.

Proof. Since $\|u\|_{\mathcal{C}} = \max\big\{\max_{\bar{\Omega}}(-u), \max_{\bar{\Omega}}(u)\big\}$, we shall bound the maxima of $\pm u$. We set $\phi(x) = \mathrm{e}^{\lambda} - \mathrm{e}^{\lambda x}$ and define two functions

$$v_{\pm}(x) = \pm u(x) - \|\mathcal{A}u\|_{\mathcal{C}}\, \phi(x).$$

Since $\mathcal{A}\phi = c\mathrm{e}^{\lambda} + (a\lambda^2 + (a' - b)\lambda - c)\mathrm{e}^{\lambda x} \geq 1$ in $\bar{\Omega}$, if $\lambda > 0$ is chosen sufficiently large, we have, with such a choice of λ,

$$\mathcal{A}v_{\pm} = \pm\mathcal{A}u - \|\mathcal{A}u\|_{\mathcal{C}}\mathcal{A}\phi \leq \pm\mathcal{A}u - \|\mathcal{A}u\|_{\mathcal{C}} \leq 0 \quad \text{in } \Omega.$$

Theorem 2.1 therefore yields

$$\begin{aligned}\max_{\bar{\Omega}}(v_{\pm}) &\leq \max\big\{v_{\pm}(0), v_{\pm}(1), 0\big\}\\ &\leq \max\big\{\pm u(0), \pm u(1), 0\big\} \leq \max\big\{|u(0)|, |u(1)|\big\},\end{aligned}$$

since $\phi \geq 0$, so that $v_{\pm} \leq \pm u \leq |u|$. Hence,

$$\begin{aligned}\max_{\bar{\Omega}}(\pm u) = \max_{\bar{\Omega}}\big(v_{\pm} + \|\mathcal{A}u\|_{\mathcal{C}}\,\phi\big) &\leq \max_{\bar{\Omega}}(v_{\pm}) + \|\mathcal{A}u\|_{\mathcal{C}}\|\phi\|_{\mathcal{C}}\\ &\leq \max\big\{|u(0)|, |u(1)|\big\} + C\|\mathcal{A}u\|_{\mathcal{C}}, \quad \text{with } C = \|\phi\|_{\mathcal{C}},\end{aligned}$$

which completes the proof. □

From Theorem 2.2 we immediately conclude the uniqueness of a solution of (2.1). In fact, if u and v were two solutions, then their difference $w = u - v$ would satisfy $\mathcal{A}w = 0$, $w(0) = w(1) = 0$, and hence $\|w\|_{\mathcal{C}} = 0$, so that $u = v$.

More generally, if u and v are two solutions of (2.1) with right hand sides f and g and boundary values u_0, u_1 and v_0, v_1, respectively, then

$$\|u - v\|_{\mathcal{C}} \leq \max\big\{|u_0 - v_0|, |u_1 - v_1|\big\} + C\|f - g\|_{\mathcal{C}}.$$

Thus the problem (2.1) is stable, i.e., a small change in data does not cause a big change in the solution.

As another application of the maximum principle we note that if all the data of the boundary value problem (2.1) are nonpositive, then the solution is nonpositive. That is, if $f \leq 0$ and $u_0, u_1 \leq 0$, then $u \leq 0$. By means of the stronger variant of the maximum principle mentioned after Theorem 2.1, we may even conclude that $u < 0$ in Ω unless $u(x) \equiv 0$. More generally, we have the following *monotonicity property*: If

$$\begin{aligned} \mathcal{A}u &= f \quad \text{in } \Omega, \qquad \text{with } u(0) = u_0,\ u(1) = u_1, \\ \mathcal{A}v &= g \quad \text{in } \Omega, \qquad \text{with } v(0) = v_0,\ v(1) = v_1, \end{aligned}$$

and if $f \leq g$, $u_0 \leq v_0$, and $u_1 \leq v_1$, then $u \leq v$.

2.2 Green's Function

We now consider the problem (2.1) with $b = 0$ and with boundary values $u_0 = u_1 = 0$. We shall derive a representation of a solution in terms of a so-called Green's function $G(x, y)$. For this purpose, let U_0 and U_1 be two solutions of the homogeneous equation such that

$$\begin{aligned} \mathcal{A}U_0 &= 0 \quad \text{in } \Omega, \qquad \text{with } U_0(0) = 1,\ U_0(1) = 0, \\ \mathcal{A}U_1 &= 0 \quad \text{in } \Omega, \qquad \text{with } U_1(0) = 0,\ U_1(1) = 1. \end{aligned}$$

To see that such solutions exist, we note that by the standard theory of ordinary differential equations the initial value problem for $\mathcal{A}u = 0$ with $u(0) = 0$, $u'(0) = 1$ has a unique solution, and that $u(1) \neq 0$ for this solution, since otherwise $u(x) \equiv 0$ in Ω by Theorem 2.2. By multiplication of this solution by an appropriate constant we obtain the desired function U_1. The function U_0 is constructed similarly, starting at $x = 1$. By Theorem 2.1 U_0 and U_1 are nonnegative. We refer to Problem 2.5 for the case when $b \neq 0$.

Theorem 2.3. *Let $b = 0$ and let U_0, U_1 be as described above. Then a solution of (2.1) with $u_0 = u_1 = 0$ is given by*

$$u(x) = \int_0^1 G(x, y) f(y)\, \mathrm{d}y, \tag{2.8}$$

where

$$G(x, y) = \begin{cases} \dfrac{1}{\kappa} U_0(x) U_1(y), & \textit{for } 0 \leq y \leq x \leq 1, \\[2ex] \dfrac{1}{\kappa} U_1(x) U_0(y), & \textit{for } 0 \leq x \leq y \leq 1, \end{cases}$$

and

$$\kappa = a(x)\big(U_0(x) U_1'(x) - U_0'(x) U_1(x)\big) \equiv \textit{constant} > 0. \tag{2.9}$$

Proof. We begin by showing that κ is constant: Since $(a\,U_j')' = c\,U_j$, we have

$$\kappa' = U_0(a\,U_1')' - U_1(a\,U_0')' = U_0\,c\,U_1 - U_1\,c\,U_0 = 0.$$

Setting $x = 0$ we find $\kappa = a(0)\,U_1'(0) \neq 0$, because otherwise $U_1(0) = U_1'(0) = 0$ and hence $U_1(x) \equiv 0$. Since U_1 is nonnegative we have $U_1'(0)$ nonnegative and hence it follows that $\kappa > 0$.

Clearly u as defined in (2.8) satisfies the homogeneous boundary conditions. To show that it is a solution of the differential equation we write

$$\begin{aligned}
u(x) &= \int_0^x G(x,y)f(y)\,\mathrm{d}y + \int_x^1 G(x,y)f(y)\,\mathrm{d}y \\
&= \frac{1}{\kappa}U_0(x)\int_0^x U_1(y)f(y)\,\mathrm{d}y + \frac{1}{\kappa}U_1(x)\int_x^1 U_0(y)f(y)\,\mathrm{d}y.
\end{aligned}$$

Hence, by differentiation,

$$\begin{aligned}
u'(x) &= \frac{1}{\kappa}\Big(U_0'(x)\int_0^x U_1(y)f(y)\,\mathrm{d}y + U_0(x)U_1(x)f(x)\Big) \\
&\quad + \frac{1}{\kappa}\Big(U_1'(x)\int_x^1 U_0(y)f(y)\,\mathrm{d}y - U_1(x)U_0(x)f(x)\Big),
\end{aligned}$$

where the terms involving $f(x)$ cancel. Multiplying by $-a(x)$ and differentiating we thus obtain, using $(a\,U_j')' = c\,U_j$ and (2.9),

$$\begin{aligned}
-(a(x)u'(x))' &= -\frac{1}{\kappa}(a(x)U_0'(x))'\int_0^x U_1(y)f(y)\,\mathrm{d}y \\
&\quad - \frac{1}{\kappa}(a(x)U_1'(x))'\int_x^1 U_0(y)f(y)\,\mathrm{d}y \\
&\quad - \frac{1}{\kappa}a(x)\Big(U_0'(x)U_1(x) - U_1'(x)U_0(x)\Big)f(x) \\
&= -\frac{1}{\kappa}c(x)U_0(x)\int_0^x U_1(y)f(y)\,\mathrm{d}y \\
&\quad - \frac{1}{\kappa}c(x)U_1(x)\int_x^1 U_0(y)f(y)\,\mathrm{d}y + f(x) \\
&= -c(x)\int_0^1 G(x,y)f(y)\,\mathrm{d}y + f(x) = -c(x)u(x) + f(x),
\end{aligned}$$

which completes the proof. □

In particular, this theorem shows the existence of a solution of the problem considered. We already know from Sect. 2.1 that the solution is unique. The representation of the solution as an integral in terms of the Green's function can also be used to obtain additional information about the solution. As a simple example we have the maximum-norm estimate

$$\|u\|_{\mathcal{C}} \leq C\|f\|_{\mathcal{C}}, \quad \text{with } C = \max_{x\in\bar{\Omega}} \int_0^1 G(x,y)\,\mathrm{d}y, \tag{2.10}$$

which gives a more precise value of the constant in Theorem 2.2. Here we have used the fact that U_0 and U_1, and hence G, are nonnegative by Theorem 2.1.

Theorem 2.3 may also be used to show the existence of a solution for general boundary values u_0 and u_1. In fact, if $\bar{u}(x) = u_0(1-x) + u_1 x$, and if v is a solution of

$$\mathcal{A}v = g := f - \mathcal{A}\bar{u} \quad \text{in } \Omega, \quad \text{with } v(0) = v(1) = 0,$$

then $u = v + \bar{u}$ satisfies $\mathcal{A}u = f$ and $u(0) = u_0,\ u(1) = u_1$.

2.3 Variational Formulation

We shall now treat our two-point boundary value problem within the framework of the Hilbert space $L_2 = L_2(\Omega)$, and derive a so-called variational formulation. We refer to App. A for the functional analytic concepts used.

We consider the boundary value problem (2.1) with homogeneous boundary conditions, i.e.,

$$\mathcal{A}u := -(au')' + bu' + cu = f \quad \text{in } \Omega = (0,1), \quad \text{with } u(0) = u(1) = 0. \tag{2.11}$$

We assume that the coefficients a, b, and c are smooth and, instead of (2.2), that

$$a(x) \geq a_0 > 0, \quad c(x) - b'(x)/2 \geq 0, \quad \text{for } x \in \bar{\Omega}. \tag{2.12}$$

Multiplying the differential equation by a function $\varphi \in \mathcal{C}_0^1 = \mathcal{C}_0^1(\Omega)$, and integrating over the interval Ω, we obtain

$$\int_0^1 \big(-(au')' + bu' + cu\big)\varphi\,\mathrm{d}x = \int_0^1 f\varphi\,\mathrm{d}x, \tag{2.13}$$

or, after integration by parts, using $\varphi(0) = \varphi(1) = 0$,

$$\int_0^1 (au'\varphi' + bu'\varphi + cu\varphi)\,\mathrm{d}x = \int_0^1 f\varphi\,\mathrm{d}x, \quad \forall \varphi \in \mathcal{C}_0^1, \tag{2.14}$$

which we refer to as the *variational* or *weak formulation* of (2.11).

Introducing the bilinear form

$$a(v,w) = \int_0^1 (av'w' + bv'w + cvw)\,\mathrm{d}x, \tag{2.15}$$

and the linear functional

$$L(w) = (f, w) = \int_0^1 f\, w \,\mathrm{d}x,$$

and using the fact that $\mathcal{C}_0^1$ is dense in $H_0^1 = H_0^1(\Omega)$, we may write the equation (2.14) as

$$a(u, \varphi) = L(\varphi), \quad \forall \varphi \in H_0^1. \tag{2.16}$$

We say that u is a *weak solution* of (2.11) if $u \in H_0^1$ and (2.16) holds. Thus we do not require a weak solution to be twice differentiable. However, if a weak solution belongs to $\mathcal{C}^2$, then it is actually a classical solution of (2.11). In fact, by integration by parts in (2.14) we conclude that (2.13) holds, i.e.,

$$\int_0^1 (\mathcal{A}u - f)\, \varphi \,\mathrm{d}x = 0, \quad \forall \varphi \in H_0^1.$$

This immediately implies $\mathcal{A}u = f$ in Ω, and since $u \in H_0^1$ we also have $u(0) = u(1) = 0$. This calculation can also be performed if $u \in H^2 \cap H_0^1$, in which case we say that u is a *strong solution* of (2.11).

We note that, with the notation of Sect. 1.2,

$$\|v\| \le \|v'\|, \quad \text{if } v(0) = v(1) = 0. \tag{2.17}$$

In fact, by the Cauchy-Schwarz inequality we have for all $x \in \Omega$,

$$|v(x)|^2 = \Big| \int_0^x v'(y) \,\mathrm{d}y \Big|^2 \le \int_0^x 1^2 \,\mathrm{d}y \int_0^x (v')^2 \,\mathrm{d}y \le x \int_0^1 (v')^2 \,\mathrm{d}y \le \|v'\|^2,$$

from which (2.17) follows by integration. This is a special case of Poincaré's inequality, which has a counterpart also for functions of several variables, see Theorem A.6. It follows at once that

$$\|v\|_1 = \big(\|v\|^2 + \|v'\|^2\big)^{1/2} \le \sqrt{2}\|v'\|, \quad \forall v \in H_0^1, \tag{2.18}$$

which shows that $\|v\|_1$ and $|v|_1 = \|v'\|$ are equivalent norms.

Using our assumption (2.12), we find that

$$\int_0^1 (bv'v + cv^2) \,\mathrm{d}x = \Big[\tfrac{1}{2} b\, v^2\Big]_0^1 + \int_0^1 (c - \tfrac{1}{2} b')\, v^2 \,\mathrm{d}x \ge 0, \quad \text{for } v \in H_0^1.$$

Hence, from (2.12) and (2.18) it follows that the bilinear form $a(v, w)$ has the property

$$a(v, v) \ge \min_{x \in \bar{\Omega}} a(x)\, \|v'\|^2 \ge \alpha \|v\|_1^2, \quad \forall v \in H_0^1, \quad \text{with } \alpha = a_0/2 > 0. \tag{2.19}$$

The inequality (2.19) expresses that the bilinear form $a(\cdot, \cdot)$ is *coercive* in H_0^1, see (A.12). Setting $\varphi = u$ in (2.16) and using (2.19) and (2.17), we find

$$\alpha\|u\|_1^2 \le a(u,u) = (f,u) \le \|f\|\,\|u\| \le \|f\|\,\|u\|_1,$$

so that

$$\|u\|_1 \le C\|f\|, \quad \text{with } C = 2/a_0. \tag{2.20}$$

The bilinear form $a(v,w)$ is also bounded on H_0^1 in the sense that (cf. (A.9))

$$|a(v,w)| \le C\|v\|_1\|w\|_1, \quad \forall v,w \in H_0^1. \tag{2.21}$$

For, estimating the coefficients in (2.15) by their maxima and using the Cauchy-Schwarz inequality, we have

$$|a(v,w)| \le C\int_0^1 \big(|v'w'| + |v'w| + |vw|\big)\,\mathrm{d}x \le C\|v\|_1\|w\|_1.$$

We now turn to the question of existence of a solution of the variational equation (2.16).

Theorem 2.4. *Assume that* (2.12) *holds and let* $f \in L_2$. *Then there exists a unique solution* $u \in H_0^1$ *of* (2.16). *This solution satisfies* (2.20).

Proof. The proof is based on the Lax-Milgram lemma, Theorem A.3. We already checked that $a(\cdot,\cdot)$ is coercive and bounded in H_0^1. The linear functional $L(\cdot)$ is also bounded in H_0^1, because

$$|L(\varphi)| = |(f,\varphi)| \le \|f\|\,\|\varphi\| \le \|f\|\,\|\varphi\|_1, \quad \forall \varphi \in H_0^1.$$

Hence the assumptions of the Lax-Milgram lemma are satisfied and it follows that there exists a unique $u \in H_0^1$ satisfying (2.16). Together with (2.20) this completes the proof. □

We remark that when $b = 0$ the bilinear form $a(\cdot,\cdot)$ is symmetric positive definite and thus an inner product, with the associated norm equivalent to $\|\cdot\|_1$. The existence of a unique solution then follows from the more elementary Riesz representation theorem, Theorem A.1.

In the symmetric case when $b = 0$, the solution of (2.16) may also be characterized as the minimizer of a certain quadratic functional, see Theorem A.2. This is a special case of the famous Dirichlet principle.

Theorem 2.5. *Assume that* (2.2) *holds and that* $b = 0$. *Let* $f \in L_2$ *and* $u \in H_0^1$ *be the solution of* (2.16), *and set*

$$F(\varphi) = \tfrac{1}{2}\int_0^1 \big(a(\varphi')^2 + c\varphi^2\big)\,\mathrm{d}x - \int_0^1 f\varphi\,\mathrm{d}x.$$

Then $F(u) \le F(\varphi)$ *for all* $\varphi \in H_0^1$, *with equality only for* $\varphi = u$.

The weak solution u of (2.16) obtained in Theorem 2.4 is actually more regular than stated there. Using our definitions one may, in fact, show that u'' exists as a weak derivative (cf. (A.21)), and that $au'' = -f + (b-a')u' + cu \in L_2$. It follows that $u \in H^2$ and that

$$a_0\|u''\| \leq \|au''\| \leq \|f\| + \|(b-a')u'\| + \|cu\| \leq \|f\| + C\|u\|_1 \leq C\|f\|.$$

Together with (2.20) this implies the *regularity estimate*

$$\|u\|_2 \leq C\|f\|. \tag{2.22}$$

We conclude that the weak solution of (2.1) found in Theorem 2.4 is actually a strong solution. The proof of H^2-regularity uses the assumption that a is smooth and $f \in L_2$. With a less smooth, or with f only in H^{-1}, see (A.30), we still obtain a weak solution in H_0^1, but then it may not belong to H^2, see Problem 2.8.

2.4 Problems

Problem 2.1. Determine explicit solutions of the boundary value problem

$$-u'' + cu = f \quad \text{in } (-1,1), \quad \text{with } u(-1) = u(1) = g,$$

where c, f, g are constants. Use this to illustrate the maximum principle.

Problem 2.2. Determine Green's functions for the following problems:

(a) $\quad -u'' = f \qquad$ in $\Omega = (0,1)$, $\qquad$ with $u(0) = u(1) = 0$,

(b) $\quad -u'' + cu = f \qquad$ in $\Omega = (0,1)$, $\qquad$ with $u(0) = u(1) = 0$,

where c is a positive constant.

Problem 2.3. Consider the nonlinear boundary value problem

$$-u'' + u = \mathrm{e}^u \quad \text{in } \Omega = (0,1), \quad \text{with } u(0) = u(1) = 0.$$

Use the maximum principle to show that all solutions are nonnegative, i.e., $u(x) \geq 0$ for all $x \in \bar{\Omega}$. Use the strong version of the maximum principle to show that all solutions are positive, i.e., $u(x) > 0$ for all $x \in \Omega$.

Problem 2.4. Assume that $b = 0$ as in Theorem 2.3 and let $G(x,y)$ be the Green's function defined there.

(a) Prove that G is symmetric, $G(x,y) = G(y,x)$.
(b) Prove that

$$a(v, G(x,\cdot)) = v(x), \quad \forall v \in H_0^1,\ x \in \Omega.$$

This means that $\mathcal{A}G(x,\cdot) = \delta_x$, where δ_x is Dirac's delta at x, defined as the linear functional $\delta_x(\phi) = \phi(x)$ for all $\phi \in \mathcal{C}_0^0$, see Problem A.9.

Problem 2.5. In the unsymmetric case when $b \neq 0$, Green's function is defined in a similar way as in Theorem 2.3:

$$G(x,y) = \begin{cases} \dfrac{U_0(x)U_1(y)}{\kappa(y)}, & \text{for } 0 \leq y \leq x \leq 1, \\[2ex] \dfrac{U_1(x)U_0(y)}{\kappa(y)}, & \text{for } 0 \leq x \leq y \leq 1. \end{cases}$$

The main difference is that κ is no longer constant. The functions U_0 and U_1 are linearly independent, and hence it follows from the theory of ordinary differential equations that their Wronski determinant $U_0U_1' - U_0'U_1$ does not vanish. As before we may then conclude that $\kappa(x) > 0$ in $\bar{\Omega}$. Repeat the steps of the proof Theorem 2.3 in this case.

Problem 2.6. Give variational formulations and prove existence of solutions of

$$-u'' = f \quad \text{in } \Omega = (0,1),$$

with the following boundary conditions

(a) $u(0) = u(1) = 0$,
(b) $u(0) = u'(1) = 0$,
(c) $-u'(0) + u(0) = u'(1) = 0$.

Problem 2.7. Consider the "beam equation" from Problem 1.3,

$$\frac{d^4u}{dx^4} = f \quad \text{in } \Omega = (0,1),$$

together with the boundary conditions

(a) $u(0) = u'(0) = u(1) = u'(1) = 0$,
(b) $u(0) = u''(0) = u(1) = u''(1) = 0$,
(c) $u(0) = u'(0) = u'(1) = u'''(1) = 0$,
(d) $u(0) = u'(0) = u''(1) = u'''(1) = 0$,
(e) $u(0) = u'(0) = u(1) = u'''(1) = 0$.

Give variational formulations and investigate existence and uniqueness of solutions of these problems. Give mechanical interpretations of the boundary conditions.

Problem 2.8. Find an explicit solution of (2.11) with $a = 1$, $b = c = 0$, and $f(x) = 1/x$. Recall from Problem A.11 that $f \in H^{-1}$ but $f \notin L_2$. Check that $u \in H_0^1$ but $u \notin H^2$. Hint: $u(x) = -x \log x$.

3 Elliptic Equations

In this chapter we study boundary value problems for elliptic partial differential equations. As we have seen in Chapt. 1 such equations are central in both theory and application of partial differential equations; they describe a large number of physical phenomena, particularly modelling stationary situations, and are stationary limits of evolution equations. After some preliminaries in Sect. 3.1 we begin by showing a maximum principle in Sect. 3.2. In the same way as for the two-point boundary value problem in Chapt. 2 this may be used to show uniqueness and continuous dependence on data for boundary value problems. In the following Sect. 3.3 we show the existence of a solution of Dirichlet's problem for Poisson's equation in a disc with homogeneous boundary conditions, using an integral representation in terms of Poisson's kernel. In Sect. 3.4 similar ideas are employed to introduce fundamental solutions of elliptic equations, and we illustrate their use by constructing a Green's function. Another important approach, presented in Sect. 3.5, is based on a variational formulation of the boundary value problem and simple functional analytic tools. In Sect. 3.6 we discuss briefly the Neumann problem, and in Sect. 3.7 we describe some regularity results.

3.1 Preliminaries

Rather than considering a general second order elliptic equation of the form (1.5) we shall restrict ourselves, for the sake of simplicity, to the special case when the matrix $A = (a_{ij})$ in (1.5) reduces to a scalar multiple $a\,I$ of the identity matrix, where a is a smooth function.

We consider first the Dirichlet problem

$$(3.1)\qquad \mathcal{A}u := -\nabla\cdot\big(a\nabla u\big) + b\cdot\nabla u + cu = f \quad \text{in } \Omega, \quad \text{with } u = g \quad \text{on } \Gamma,$$

where $\Omega \subset \mathbf{R}^d$ is a domain with appropriately smooth boundary Γ, where the coefficients $a = a(x)$, $b = b(x)$, $c = c(x)$ are smooth and such that

$$(3.2)\qquad a(x) \geq a_0 > 0, \quad c(x) \geq 0, \quad \forall x \in \Omega,$$

and where f and g are given functions on Ω and Γ, respectively. This is the stationary case of the heat equation (1.18).

The particular case $a = 1$, $b = 0$, $c = 0$ is Poisson's equation, i.e.,

$$-\Delta u := -\sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2} = f. \tag{3.3}$$

When $f = 0$ this equation is referred to as Laplace's equation and its solutions are called harmonic functions.

We note that if v and w are solutions of the two problems

$$\begin{aligned} \mathcal{A}v &= 0 \quad \text{in } \Omega, \qquad \text{with } v = g \quad \text{on } \Gamma, \\ \mathcal{A}w &= f \quad \text{in } \Omega, \qquad \text{with } w = 0 \quad \text{on } \Gamma, \end{aligned}$$

then $u = v + w$ is a solution of (3.1). It is therefore sometimes convenient to consider separately the homogeneous equation with given boundary values and the inhomogeneous equation with vanishing boundary values.

One may also study the partial differential equation in (3.1) together with Robin's boundary condition

$$a\frac{\partial u}{\partial n} + h\big(u - g\big) = 0 \quad \text{on } \Gamma, \tag{3.4}$$

where the coefficient $h = h(x)$ is positive and n is the outward unit normal to Γ. The Dirichlet boundary condition used in (3.1) may be formally obtained as the extreme case $h = \infty$ of (3.4). At the other extreme, $h = 0$, we obtain Neumann's boundary condition

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma.$$

Sometimes one considers mixed boundary conditions in which, e.g., Dirichlet boundary conditions are given on one part of the boundary and Neumann conditions on the remaining part. A function $u \in \mathcal{C}^2(\bar{\Omega})$ that satisfies the differential equation and the boundary condition in (3.1) is called a *classical solution* of this boundary value problem.

3.2 A Maximum Principle

We begin our study of the Dirichlet problem (3.1) by showing a maximum principle analogous to that of Theorem 2.1.

Theorem 3.1. *Consider the differential operator $\mathcal{A}$ in (3.1), and assume that $u \in \mathcal{C}^2 = \mathcal{C}^2(\bar{\Omega})$ and*

$$\mathcal{A}u \le 0 \quad \big(\mathcal{A}u \ge 0\big) \quad \textit{in } \Omega. \tag{3.5}$$

(i) *If $c = 0$, then*

$$\max_{\bar{\Omega}} u = \max_{\Gamma} u \quad \Big(\min_{\bar{\Omega}} u = \min_{\Gamma} u \Big). \tag{3.6}$$

(ii) *If $c \geq 0$ in Ω, then*

$$\max_{\bar{\Omega}} u \leq \max \big\{ \max_{\Gamma} u, 0 \big\} \quad \Big(\min_{\bar{\Omega}} u \geq \min \big\{ \min_{\Gamma} u, 0 \big\} \Big). \tag{3.7}$$

Proof. (i) Let ϕ be a function such that $\phi \geq 0$ in $\bar{\Omega}$ and $\mathcal{A}\phi < 0$ in Ω. Such a function is, e.g., $\phi(x) = \mathrm{e}^{\lambda x_1}$ for λ so large that $\mathcal{A}\phi = (-a\,\lambda^2 + (b_1 - \partial a/\partial x_1)\lambda)\mathrm{e}^{\lambda x_1} < 0$ in $\bar{\Omega}$. Assume now that u attains its maximum at an interior point x_0 in Ω but not on Γ. Then for ϵ sufficiently small this is true also for $v = u + \epsilon\phi$. But $\mathcal{A}v = \mathcal{A}u + \epsilon\mathcal{A}\phi < 0$ in $\bar{\Omega}$. On the other hand, if the maximum of v is $v(\bar{x}_0)$, then $\nabla v(\bar{x}_0) = 0$ and hence $\mathcal{A}v(\bar{x}_0) = -a(\bar{x}_0)\Delta v(\bar{x}_0) \geq 0$, which is a contradiction, and thus shows our claim.

(ii) If $u \leq 0$ in Ω, then (3.7) holds trivially. Otherwise assume that $\max_{\bar{\Omega}} u = u(x_0) > 0$ and $x_0 \in \Omega$. Let Ω_0 be the largest open connected subset of Ω containing x_0 in which $u > 0$. We now have $\tilde{\mathcal{A}}u := \mathcal{A}u - cu \leq 0$ in Ω_0. Part (i), applied with the operator $\tilde{\mathcal{A}}$ in Ω_0, therefore implies $u(x_0) = \max_{\Gamma_0} u$, where Γ_0 is the boundary of Ω_0. But then Γ_0 could not lie completely in the open set Ω, for then there would be a point on Γ_0 where u were positive, and Ω_0 would not be as large as possible with $u > 0$. This shows (3.7). □

Theorem 3.1 implies stability with respect to the maximum-norm.

Theorem 3.2. *Let $u \in \mathcal{C}^2(\bar{\Omega})$. Then there is a constant C such that*

$$\|u\|_{\mathcal{C}(\bar{\Omega})} \leq \|u\|_{\mathcal{C}(\Gamma)} + C\|\mathcal{A}u\|_{\mathcal{C}(\bar{\Omega})}.$$

Proof. Let ϕ be a function such that $\phi \geq 0$ and $\mathcal{A}\phi \leq -1$ in Ω, e.g., a suitable multiple of the function ϕ in the proof of Theorem 3.1. We now define two functions $v_{\pm}(x) = \pm u(x) + \|\mathcal{A}u\|_{\mathcal{C}(\bar{\Omega})}\phi(x)$. Then

$$\mathcal{A}v_{\pm} = \pm\mathcal{A}u + \|\mathcal{A}u\|_{\mathcal{C}(\bar{\Omega})}\mathcal{A}\phi \leq 0, \quad \text{in } \Omega.$$

Therefore both functions $v_{\pm}$ take their maxima on Γ, so that

$$\begin{aligned} v_{\pm}(x) \leq \max_{\Gamma}(v_{\pm}) &\leq \max_{\Gamma}(\pm u) + \|\mathcal{A}u\|_{\mathcal{C}(\bar{\Omega})}\|\phi\|_{\mathcal{C}(\Gamma)} \\ &\leq \|u\|_{\mathcal{C}(\Gamma)} + C\|\mathcal{A}u\|_{\mathcal{C}(\bar{\Omega})}, \quad \text{with } C = \|\phi\|_{\mathcal{C}(\Gamma)}. \end{aligned}$$

Since $\pm u(x) \leq v_{\pm}(x)$ this proves the theorem. □

In the same way as for the two-point boundary value problem it follows that there is at most one solution of our Dirichlet problem (3.1), and that, if $u_j,\ j = 1, 2$, are solutions of (3.1) with $f = f_j,\ g = g_j,\ j = 1, 2$, then

$$\|u_1 - u_2\|_{\mathcal{C}(\bar{\Omega})} \leq \|g_1 - g_2\|_{\mathcal{C}(\Gamma)} + C\|f_1 - f_2\|_{\mathcal{C}(\bar{\Omega})}.$$

3.3 Dirichlet's Problem for a Disc. Poisson's Integral

In this section we study the Dirichlet problem to find a harmonic function in a disc $\Omega = \{x \in \mathbf{R}^2 : |x| < R\}$ with given boundary values, i.e.,

$$
\begin{aligned}
-\Delta u &= 0, && \text{for } |x| < R, \\
u(R\cos\varphi, R\sin\varphi) &= g(\varphi), && \text{for } 0 \le \varphi < 2\pi.
\end{aligned} \tag{3.8}
$$

In the following theorem a solution of (3.8) is given as an integral over the boundary of the disc.

Theorem 3.3. (Poisson's integral formula.) *Let $P_R(r,\varphi)$ denote the Poisson kernel*

$$P_R(r,\varphi) = \frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos\varphi}.$$

Then, using polar coordinates $x = (r\cos\varphi, r\sin\varphi)$, the function defined by

$$u(x) = \frac{1}{2\pi}\int_0^{2\pi} P_R(r, \varphi - \psi) g(\psi)\,\mathrm{d}\psi, \tag{3.9}$$

is a solution of (3.8) *for g appropriately smooth,*

Proof. We first note that, for each $n \ge 0$, $v(x) = r^n \mathrm{e}^{\pm\mathrm{i}n\varphi}$ is a harmonic function. In fact, we have

$$
\begin{aligned}
\Delta v &= \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \varphi^2}, \\
&= \Big(n(n-1)r^{n-2} + \frac{1}{r} n r^{n-1} - \frac{1}{r^2} n^2 r^n\Big)\mathrm{e}^{\pm\mathrm{i}n\varphi} = 0.
\end{aligned}
$$

It follows, for c_n bounded, say, that the series

$$u(x) = \sum_{n=-\infty}^{\infty} c_n \Big(\frac{r}{R}\Big)^{|n|} \mathrm{e}^{\mathrm{i}n\varphi} \tag{3.10}$$

is harmonic in Ω. We assume now that $g(\varphi)$ has a Fourier series

$$g(\varphi) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}n\varphi}.$$

which is absolutely convergent. The function $u(x)$ in (3.10) with the coefficients c_n is a then solution of (3.8), and u is continuous in $\bar{\Omega}$. The latter means that $u(r\mathrm{e}^{\mathrm{i}\psi}) \to g(\mathrm{e}^{\mathrm{i}\varphi})$ when $r \to R$, $\psi \to \varphi$. To see that this holds, we choose N so large that $\sum_{|n|>N} |c_n| < \epsilon/3$ and write

$$|u(r\mathrm{e}^{\mathrm{i}\psi}) - g(\mathrm{e}^{\mathrm{i}\varphi})| \le \sum_{|n|\le N} |c_n| \left| \Big(\frac{r}{R}\Big)^{|n|} \mathrm{e}^{\mathrm{i}n\psi} - \mathrm{e}^{\mathrm{i}n\varphi} \right| + 2\sum_{|n|>N} |c_n|.$$

Here obviously the first term on the right tends to 0 when $r \to R$, $\psi \to \varphi$, and hence becomes smaller than $\epsilon/3$, which shows our claim.

Recall that the Fourier coefficients of g are given by

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}n\psi} g(\psi)\, \mathrm{d}\psi.$$

Formally we thus have

$$u(x) = \frac{1}{2\pi} \int_0^{2\pi} \sum_{n=-\infty}^{\infty} \left(\frac{r}{R}\right)^{|n|} \mathrm{e}^{\mathrm{i}n(\varphi-\psi)} g(\psi)\, \mathrm{d}\psi,$$

which is of the form (3.9) with

$$P_R(r, \varphi) = \sum_{n=-\infty}^{\infty} \left(\frac{r}{R}\right)^{|n|} \mathrm{e}^{\mathrm{i}n\varphi}.$$

Setting $z = (r/R)\mathrm{e}^{\mathrm{i}\varphi}$ we have

$$\begin{aligned} P_R(r, \varphi) &= 1 + 2\,\mathrm{Re} \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^{n} \mathrm{e}^{\mathrm{i}n\varphi} \\ &= 2\,\mathrm{Re} \sum_{n=0}^{\infty} z^n - 1 = \mathrm{Re}\, \frac{2}{1-z} - 1 = \mathrm{Re}\, \frac{1+z}{1-z} \\ &= \mathrm{Re}\, \frac{R + r\mathrm{e}^{\mathrm{i}\varphi}}{R - r\mathrm{e}^{\mathrm{i}\varphi}} = \frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos\varphi}, \end{aligned}$$

which completes the proof. □

One consequence of the theorem is that if u is a harmonic function in Ω, $\tilde{x}$ is any point in Ω, and if the disc $\{x : |x - \tilde{x}| \le R\}$ is contained in Ω, then

$$u(\tilde{x}) = \frac{1}{2\pi} \int_0^{2\pi} u(\tilde{x}_1 + R\cos\psi, \tilde{x}_2 + R\sin\psi)\, \mathrm{d}\psi, \tag{3.11}$$

since $P_R(0, \varphi) = 1$. Hence $u(\tilde{x})$ is the average of the values of $u(x)$ with $|x - \tilde{x}| = R$. Thus the value of u at the center of a disc equals the average of its boundary values. We say that u satisfies the meanvalue property. This proves a special case of the strong maximum principle we have mentioned earlier: If a harmonic function u takes its maximum value at an interior point of Ω, then it is constant. In fact, if $\tilde{x}$ is an interior point of Ω where u attains its maximum, then by (3.11) $u(x) = u(\tilde{x})$ for all x with $\{x : |x - \tilde{x}| = R\} \subset \Omega$, and since R is arbitrary and Ω connected it follows easily that u takes the constant value $u(\tilde{x})$ in $\bar{\Omega}$. In particular, the maximum is also attained on Γ.

3.4 Fundamental Solutions. Green's Function

Let u be a solution of the inhomogeneous equation

$$\mathcal{A}u = f \quad \text{in } \mathbf{R}^d, \tag{3.12}$$

where $\mathcal{A}$ is as in (3.1), with $b = 0$. Multiplying by $\varphi \in \mathcal{C}_0^\infty(\mathbf{R}^d)$, integrating over $\mathbf{R}^d$, and integrating by parts twice, we obtain

$$(u, \mathcal{A}\varphi) = (f, \varphi) = \int_{\mathbf{R}^d} f(x)\,\varphi(x)\,\mathrm{d}x, \quad \forall \varphi \in \mathcal{C}_0^\infty(\mathbf{R}^d). \tag{3.13}$$

We say that U is a fundamental solution of (3.12) if U is smooth for $x \neq 0$, has a singularity at $x = 0$ such that $U \in L_1(B)$, where $B = \{x \in \mathbf{R}^d : |x| < 1\}$, and

$$|D^\alpha U(x)| \le C_\alpha |x|^{2-d-|\alpha|} \quad \text{for } |\alpha| \neq 0, \tag{3.14}$$

and if

$$(U, \mathcal{A}\varphi) = \varphi(0), \quad \forall \varphi \in \mathcal{C}_0^\infty(\mathbf{R}^d). \tag{3.15}$$

This means that, in the sense of weak derivative (see (A.21)),

$$\mathcal{A}U = \delta,$$

where δ is Dirac's delta, defined in Problem A.9.

We now use the fundamental solution to construct a solution to (3.12).

Theorem 3.4. *If U is a fundamental solution of* (3.12) *and if $f \in \mathcal{C}_0^1(\mathbf{R}^d)$, then*

$$u(x) = (U * f)(x) = \int_{\mathbf{R}^d} U(x-y) f(y)\,\mathrm{d}y$$

is a solution of (3.12).

Proof. We have, by (3.15),

$$\int_{\mathbf{R}^d} U(x-y)\mathcal{A}\varphi(x)\,\mathrm{d}x = \int_{\mathbf{R}^d} U(z)\mathcal{A}\varphi(z+y)\,\mathrm{d}z = (U, \mathcal{A}\varphi(\cdot + y)) = \varphi(y).$$

Hence, if $u = U * f$, then, by changing the order of integration,

$$\begin{aligned} (u, \mathcal{A}\varphi) &= \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} U(x-y) f(y)\,\mathrm{d}y\, \mathcal{A}\varphi(x)\,\mathrm{d}x \\ &= \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} U(x-y)\mathcal{A}\varphi(x)\,\mathrm{d}x\, f(y)\,\mathrm{d}y \\ &= \int_{\mathbf{R}^d} \varphi(y)\, f(y)\,\mathrm{d}y = (f, \varphi). \end{aligned} \tag{3.16}$$

Since $f \in \mathcal{C}_0^1$ it follows that $u \in \mathcal{C}^2$ because with $D_i = \partial/\partial x_i$ we have $D_i D_j u(x) = (D_i U * D_j f)(x)$ (cf. App. A.3) and $D_i U \in L_1(\mathbf{R}^d)$ and $D_j f \in \mathcal{C}_0(\mathbf{R}^d)$. Thus we may integrate by parts in (3.16) to obtain (cf. (3.13))

$$(\mathcal{A}u - f, \varphi) = 0, \quad \forall \varphi \in \mathcal{C}_0^\infty(\mathbf{R}^d),$$

from which we conclude that $\mathcal{A}u = f$. □

In the next theorem we determine fundamental solutions for Poisson's equation in two and three dimensions.

Theorem 3.5. *Let*

$$U(x) = \begin{cases} -\dfrac{1}{2\pi} \log |x|, & \textit{when } d = 2, \\[2ex] \dfrac{1}{4\pi |x|}, & \textit{when } d = 3. \end{cases}$$

Then U is a fundamental solution for Poisson's equation (3.3).

Proof. We carry out the proof for $d = 2$; the proof for $d = 3$ is similar. By differentiation we find, for $x \neq 0$,

$$-\frac{\partial U}{\partial x_j} = \frac{1}{2\pi} \frac{x_j}{|x|^2}, \qquad -\frac{\partial^2 U}{\partial x_j^2} = \frac{1}{2\pi} \frac{|x|^2 - 2x_j^2}{|x|^4},$$

so that, in particular, $-\Delta U = 0$ for $x \neq 0$. Similarly, (3.14) holds.

Let $\varphi \in \mathcal{C}_0^\infty(\mathbf{R}^2)$. We have by Green's formula, with $n = x/|x|$,

$$\int_{|x|>\epsilon} U(-\Delta\varphi) \, \mathrm{d}x = \int_{|x|>\epsilon} (-\Delta U)\varphi \, \mathrm{d}x - \int_{|x|=\epsilon} \Big(\varphi \frac{\partial U}{\partial n} - \frac{\partial \varphi}{\partial n} U\Big) \, \mathrm{d}s.$$

Note that n points inwards here. The first term on the right side vanishes. Further, since

$$\frac{\partial U}{\partial n} = \frac{x_1}{|x|} \frac{\partial U}{\partial x_1} + \frac{x_2}{|x|} \frac{\partial U}{\partial x_2} = \frac{1}{2\pi} \frac{1}{|x|} = \frac{1}{2\pi\epsilon}, \quad \text{for } |x| = \epsilon,$$

we have

$$\int_{|x|=\epsilon} \varphi \frac{\partial U}{\partial n} \, \mathrm{d}s = \frac{1}{2\pi\epsilon} \int_{|x|=\epsilon} \varphi \, \mathrm{d}s \to \varphi(0), \quad \text{as } \epsilon \to 0.$$

Also,

$$\Big| \int_{|x|=\epsilon} \frac{\partial \varphi}{\partial n} U \, \mathrm{d}s \Big| = \Big| \frac{1}{2\pi} \log(\epsilon) \int_{|x|=\epsilon} \frac{\partial \varphi}{\partial n} \, \mathrm{d}s \Big| \le \epsilon |\log(\epsilon)| \, \|\nabla\varphi\|_{\mathcal{C}} \to 0, \text{ as } \epsilon \to 0.$$

Hence

$$(U, (-\Delta)\varphi) = \lim_{\epsilon \to 0} \int_{|x|>\epsilon} U(x)(-\Delta)\varphi(x) \, \mathrm{d}x = \varphi(0).$$

□

We may now construct a Green's function for the boundary value problem

$$(3.17) \qquad -\Delta u = f \quad \text{in } \Omega, \quad \text{with } u = 0 \quad \text{on } \Gamma,$$

namely a function $G(x, y)$ defined for $x, y \in \Omega$ such that the solution of (3.17) may be represented as

$$(3.18) \qquad u(x) = \int_\Omega G(x, y) f(y)\, \mathrm{d}y.$$

Let

$$(3.19) \qquad G(x, y) = U(x - y) - v_y(x),$$

where U is the fundamental solution for $-\Delta$ from Theorem 3.5 and, for fixed $y \in \Omega$, let v_y be the solution of

$$-\Delta_x v_y(x) = 0 \quad \text{in } \Omega, \quad \text{with } v_y(x) = U(x - y) \quad \text{on } \Gamma.$$

In the next section we shall show that this problem has a solution. The Green's function thus has the singularity of the fundamental solution and vanishes for $x \in \Gamma$, and it is easily seen that the function defined by (3.18) is therefore a solution of (3.17). It is also the only solution, because we have already proved uniqueness in Sect. 3.2. Note that $G(x, y)$ consists of a singular part, $U(x - y)$ with a singularity at $x = y$, and a smooth part, $v_y(x)$.

3.5 Variational Formulation of the Dirichlet Problem

We first consider the Dirichlet problem with homogeneous boundary conditions

$$(3.20) \quad \mathcal{A}u := -\nabla \cdot \big(a \nabla u\big) + b \cdot \nabla u + cu = f \quad \text{in } \Omega, \quad \text{with } u = 0 \quad \text{on } \Gamma,$$

where the coefficients a, b, and c are smooth functions in $\bar{\Omega}$ which satisfy

$$(3.21) \qquad a(x) \geq a_0 > 0, \quad c(x) - \tfrac{1}{2} \nabla \cdot b(x) \geq 0, \quad \text{for } x \in \Omega,$$

and where f is a given function. In the classical formulation of this problem one looks for a function $u \in \mathcal{C}^2 = \mathcal{C}^2(\bar{\Omega})$ which satisfies (3.20). In this section we shall reformulate (3.20) in variational form and seek a solution in the larger class H_0^1. In some cases it is then possible to prove such regularity for this solution that it is indeed a classical solution.

Assuming first that u is a solution in $\mathcal{C}^2$, we multiply (3.20) by $v \in \mathcal{C}_0^1$ and integrate over Ω. By Green's formula and since $v = 0$ on Γ, we find that

$$(3.22) \quad \int_\Omega f v \, \mathrm{d}x = \int_\Omega \mathcal{A}u \, v \, \mathrm{d}x = \int_\Omega (a \, \nabla u \cdot \nabla v + b \cdot \nabla u \, v + c \, u \, v) \, \mathrm{d}x \quad \forall v \in \mathcal{C}_0^1,$$

and then also, since $\mathcal{C}_0^1$ is dense in H_0^1,

$$\int_\Omega \big(a\,\nabla u \cdot \nabla v + b \cdot \nabla u\, v + c\, u\, v\big)\,\mathrm{d}x = \int_\Omega f v\,\mathrm{d}x, \quad \forall v \in H_0^1. \tag{3.23}$$

The variational problem corresponding to (3.20) is thus to find $u \in H_0^1$ such that (3.23) holds. It will be shown below, by means of the Lax-Milgram lemma, that this problem admits a unique solution for $f \in L_2$. We say that this solution is a *weak* or *variational solution* of (3.20).

We have thus seen that a classical solution is also a weak solution. Conversely, suppose that $u \in H_0^1$ is a weak solution, i.e., u satisfies (3.23). If *in addition* we know that $u \in \mathcal{C}^2$, then by Green's formula we have from (3.23)

$$\int_\Omega f v\,\mathrm{d}x = \int_\Omega \big(a\,\nabla u \cdot \nabla v + b \cdot \nabla u\, v + c\, u\, v\big)\,\mathrm{d}x = \int_\Omega \mathcal{A}\, u\, v\,\mathrm{d}x, \quad \forall v \in H_0^1,$$

i.e.,

$$\int_\Omega (\mathcal{A}u - f)v\,\mathrm{d}x = 0, \quad \forall v \in H_0^1.$$

If $f \in \mathcal{C}$ we have $\mathcal{A}u - f \in \mathcal{C}$, and therefore this relation implies

$$\mathcal{A}u(x) - f(x) = 0, \quad \forall x \in \Omega.$$

Because $u \in H_0^1$ we also have $u = 0$ on Γ, and it follows that u is a classical solution of (3.20). A weak solution which is smooth enough is thus also a classical solution. However, depending on the data f and the domain Ω, a weak solution may or may not be smooth enough to be a classical solution and the weak formulation (3.23) therefore really constitutes an extension of the classical formulation. Note that the weak formulation (3.23) is meaningful for any $f \in L_2$, so that, e.g., f may be discontinuous, while the classical formulation (3.20) requires f to be continuous. If $f \in L_2$ and $u \in H^2 \cap H_0^1$ satisfies (3.20), then we say that u is a *strong solution.* Clearly, a classical solution is also a strong solution, and a strong solution is a weak solution. Further a weak solution that belongs to H^2 is a strong solution. We shall return below to the problem of the regularity of weak solutions.

We are now ready to show the existence of a weak solution. We use our standard notation from Sect. 1.2.

Theorem 3.6. *Assume that* (3.21) *holds and let* $f \in L_2$. *Then the boundary value problem* (3.20) *admits a unique weak solution, i.e., there exists a unique* $u \in H_0^1$ *which satisfies* (3.23). *Moreover, there exists a constant* C *independent of* f *such that*

$$|u|_1 \le C\|f\|. \tag{3.24}$$

Proof. We apply the Lax-Milgram lemma, Theorem A.3, in the Hilbert space $V = H_0^1$ equipped with the norm $|\cdot|_1$, and with

$$(3.25)\quad a(v,w) = \int_\Omega \big(a\nabla v\cdot\nabla w + b\cdot\nabla v\, w + c\, v\, w\big)\,\mathrm{d}x \quad \text{and} \quad L(v) = \int_\Omega f v\,\mathrm{d}x.$$

Clearly the bilinear form $a(\cdot,\cdot)$ is bounded in H_0^1 and it is coercive when (3.21) holds, since

$$a(v,v) = \int_\Omega \big(a|\nabla v|^2 + (c - \tfrac{1}{2}\nabla\cdot b)|v|^2\big)\,\mathrm{d}x \geq a_0|v|_1^2, \quad \forall v \in H_0^1.$$

Further $L(\cdot)$ is a bounded linear functional on H_0^1, since by Poincaré's inequality, Theorem A.6,

$$|L(v)| \leq \|f\|\,\|v\| \leq C\|f\|\,|v|_1.$$

This implies that $\|L\|_{V^*} \leq C\|f\|$ and the statement of the theorem thus follows directly from Theorem A.3. □

We observe that when $b = 0$, (3.21) reduces to (3.2), and the bilinear form $a(\cdot,\cdot)$ is an inner product on H_0^1. The theorem can then be proved by means of the Riesz representation theorem. In this case Theorem A.2 shows that the weak solution of (3.20) may also be characterized as follows:

Theorem 3.7. (Dirichlet's principle.) *Assume that* (3.2) *holds and that* $b = 0$. *Let* $f \in L_2$ *and* $u \in H_0^1$ *be the solution of* (3.23), *and set*

$$(3.26)\qquad F(v) = \tfrac{1}{2}\int_\Omega \big(a|\nabla v|^2 + c\,v^2\big)\,\mathrm{d}x - \int_\Omega f v\,\mathrm{d}x.$$

Then $F(u) \leq F(v)$ *for all* $v \in H_0^1$, *with equality only for* $v = u$.

Remark 3.1. If (3.20) is considered, e.g., to be a model of an elastic membrane fixed at its boundary, then $F(v)$ as defined by (3.26) is the *potential energy* associated with the deflection v; the first term in $F(v)$ corresponds to the *internal elastic energy* and the second term is a *load potential* (analogous interpretations can be made for other problems in mechanics and physics that are modeled by (3.20)). Dirichlet's principle in this case corresponds to the *Principle of Minimum Potential Energy* in mechanics and (3.23) to the *Principle of Virtual Work.*

We now consider the boundary value problem with inhomogeneous boundary condition,

$$(3.27)\qquad \mathcal{A}u = f \quad \text{in } \Omega, \quad \text{with } u = g \quad \text{on } \Gamma,$$

where we assume that $f \in L_2$ and $g \in L_2(\Gamma)$. The weak formulation of this problem is then to find $u \in H^1$ such that, with $a(\cdot,\cdot)$ and $L(\cdot)$ as in (3.25),

$$(3.28)\qquad a(u,v) = L(v), \quad \forall v \in H_0^1, \quad \text{with } \gamma\, u = g,$$

where $\gamma : H^1 \to L_2(\Gamma)$ is the trace operator, cf. Theorem A.4. For the existence of a solution, we assume that the given function g on Γ is the trace of some function $u_0 \in H^1$, i.e., $g = \gamma u_0$. Setting $w = u - u_0$, we then seek $w \in H_0^1$ satisfying

$$a(w, v) = L(v) - a(u_0, v), \quad \forall v \in H_0^1. \tag{3.29}$$

The right hand side is a bounded linear functional on H_0^1 and hence it follows by the Lax-Milgram lemma that there exists a unique $w \in H_0^1$ satisfying (3.29). Clearly, $u = u_0 + w$ satisfies (3.28) and $\gamma u = g$. This solution is unique, for if (3.27) had two weak solutions u_1, u_2 with the same data f, g, then their difference $u_1 - u_2 \in H_0^1$ would be a weak solution of (3.20) with $f = 0$, and hence the stability estimate (3.24) would imply $u_1 - u_2 = 0$, i.e., $u_1 = u_2$. Hence, (3.27) has a unique weak solution. In particular, the solution u is independent of the choice of extension u_0 of the boundary values g.

When $b = 0$, the weak solution $u \in H^1$ can equivalently be characterized as the unique solution of the minimization problem

$$\inf_{\substack{v \in H^1 \\ \gamma v = g}} \Big(\tfrac{1}{2} \int_\Omega \big(a|\nabla v|^2 + c\,v^2\big)\,\mathrm{d}x - \int_\Omega f v\,\mathrm{d}x \Big).$$

3.6 A Neumann Problem

We now consider the Neumann problem

$$\mathcal{A}u := -\nabla \cdot \big(a\nabla u\big) + cu = f \quad \text{in } \Omega, \quad \text{with } \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma, \tag{3.30}$$

where we now in addition to (3.2) require $c(x) \geq c_0 > 0$ in Ω, and where $f \in L_2$. (The case $c = 0$ is discussed in Problem 3.9.) For a variational formulation of (3.30) we multiply the differential equation in (3.30) by $v \in \mathcal{C}^1$ (note that we do not require v to satisfy any boundary conditions), and integrate over Ω using Green's formula, to obtain

$$\int_\Omega f v\,\mathrm{d}x = \int_\Omega \mathcal{A}u\,v\,\mathrm{d}x = -\int_\Gamma a\frac{\partial u}{\partial n} v\,\mathrm{d}s + \int_\Omega \big(a\nabla u \cdot \nabla v + c\,uv\big)\,\mathrm{d}x,$$

so that since $\partial u/\partial n = 0$ on Γ,

$$\int_\Omega \big(a\nabla u \cdot \nabla v + c\,uv\big)\,\mathrm{d}x = \int_\Omega f v\,\mathrm{d}x, \quad \forall v \in \mathcal{C}^1. \tag{3.31}$$

Conversely, if $u \in \mathcal{C}^2$ satisfies (3.31), then by Green's formula we have

$$\int_\Omega \big(\mathcal{A}u - f\big) v\,\mathrm{d}x + \int_\Gamma a\frac{\partial u}{\partial n} v\,\mathrm{d}s = 0, \quad \forall v \in \mathcal{C}^1. \tag{3.32}$$

If we first let v vary only over $\mathcal{C}_0^1$, we see that u must satisfy the differential equation in (3.30). Thus, the first term on the left-hand side of (3.32) vanishes, and by varying v on Γ, we see that u also satisfies the boundary condition in (3.30).

We are thus led to the following variational formulation of (3.30): Find $u \in H^1$ such that

$$a(u,v) = L(v), \quad \forall v \in H^1, \tag{3.33}$$

where $a(\cdot,\cdot)$ and $L(\cdot)$ are as in (3.25) with $b = 0$.

We have seen that if u is a classical solution of (3.30), then u satisfies (3.33). Conversely, if u satisfies (3.33) and in addition $u \in \mathcal{C}^2$, then u is a classical solution of (3.30).

By the Riesz representation theorem we have this time the following existence, uniqueness, and stability result. Note that since $c(x) \geq c_0 > 0$ the bilinear form $a(\cdot,\cdot)$ is an inner product on H^1.

Theorem 3.8. *If $f \in L_2$, then the Neumann problem* (3.30) *admits a unique weak solution, i.e., there is a unique function $u \in H^1$ that satisfies* (3.33). *Moreover,*

$$\|u\|_1 \leq C\|f\|.$$

Remark 3.2. Note that the Neumann boundary condition $\partial u/\partial n = 0$ on Γ is not enforced explicitly in the variational formulation (3.33); the function u is just required to belong to H^1. The boundary condition is implicitly contained in (3.33), since the test function v may be an arbitrary function in H^1. Such a boundary condition, which does not have to be enforced explicitly, is called a *natural boundary condition.* In contrast, a boundary condition, such as the Dirichlet condition $u = g$ on Γ, which is imposed explicitly as part of the variational formulation, is said to be an *essential boundary condition.*

Remark 3.3. The problem

$$\mathcal{A}u = f \quad \text{in } \Omega, \quad \text{with } a\frac{\partial u}{\partial n} = g \quad \text{on } \Gamma, \tag{3.34}$$

where $f \in L_2(\Omega)$ and $g \in L_2(\Gamma)$ can be given the variational formulation: Find $u \in H^1$ such that

$$a(u,v) = L(v), \quad \forall v \in H^1, \tag{3.35}$$

where $a(\cdot,\cdot)$ is as in (3.25) with $b = 0$ and

$$L(v) = \int_\Omega f v \,\mathrm{d}x + \int_\Gamma g v \,\mathrm{d}s.$$

By the Cauchy-Schwarz inequality and the trace inequality (Theorem A.4) we have

$$|L(v)| \leq \|f\| \, \|v\| + \|g\|_{L_2(\Gamma)} \|v\|_{L_2(\Gamma)} \leq \big(\|f\| + C\|g\|_{L_2(\Gamma)}\big) \|v\|_1,$$

and thus $L(\cdot)$ is a bounded linear form on H^1. The Riesz representation theorem therefore yields the existence and uniqueness of a function $u \in H^1$ satisfying (3.35). See also Problem 3.7.

3.7 Regularity

We have learned in Theorem 3.6 that for any $f \in L_2$ the Dirichlet problem (3.20) has unique weak solution $u \in H_0^1$. It can be proved that if Γ is smooth, or if Γ is a convex polygon, then, in fact, $u \in H^2$, and there is a constant C independent of f such that

$$\|u\|_2 \leq C\|f\|.$$

Since $f = \mathcal{A}u$, this may also be expressed as

$$\|u\|_2 \leq C\|\mathcal{A}u\|, \quad \forall u \in H^2 \cap H_0^1. \tag{3.36}$$

Note that, when applied with, e.g., $\mathcal{A} = -\Delta$, this inequality means that it is possible to estimate the L_2-norm of *all* second order derivatives of a function u, which vanishes on Γ, in terms of the L_2-norm of the special combination of second derivates of u given by the Laplacian $-\Delta$. We refer to Problem 3.10 for an example, with Ω neither smooth nor convex, for which the regularity estimate (3.36) does not hold.

The inequality (3.36) shows that u and its first and second order derivatives depend continuously on f in the sense that if u_1 and u_2 satisfy

$$-\mathcal{A}u_i = f_i \quad \text{in } \Omega, \quad \text{with } u_i = 0 \quad \text{on } \Gamma, \quad \text{for } i = 1, 2,$$

then

$$\Big(\sum_{|\alpha| \leq 2} \|D^\alpha u_1 - D^\alpha u_2\|^2\Big)^{1/2} \leq C\|f_1 - f_2\|.$$

If Γ is smooth, then (3.36) can be generalized as follows. For any integer $k \geq 0$ there is a constant C independent of f such that if u is the weak solution of (3.20) with $f \in H^k$, then $u \in H^{k+2} \cap H_0^1$ and

$$\|u\|_{k+2} \leq C\|f\|_k. \tag{3.37}$$

In particular, in view of Sobolev's inequality, Theorem A.5, this implies that if $k > d/2$, then $u \in \mathcal{C}^2$ and thus u is also a classical solution of (3.20).

When Γ is a polygon the situation is not so favorable. In fact, if $\mathcal{A} = -\Delta$ and $\Omega \subset \mathbf{R}^2$ has a corner with interior angle ω, then using polar coordinates (r, φ) centered at the corner, with $\varphi = 0$ corresponding to one of the edges, one can show that the solution of (3.20) behaves as $u(r, \varphi) = c\, r^\beta \sin(\beta\varphi)$ near the corner, with $\beta = \pi/\omega$. For such a function to have H^k-regularity near the corner, it is necessary that $(\partial/\partial r)^k u(r,\varphi) \in L_2(\Omega_0)$, where $\Omega_0 \subset \Omega$ contains a neighborhood of the corner under consideration, but no other corners. But this requires that

$$\big(\beta(\beta-1)\cdots(\beta-k+1)\big)^2 \int_0^b r^{2(\beta-k)} r \, \mathrm{d}r < \infty$$

for b sufficiently small, or that $2(\beta-k)+1 \geq -1$ (note that $\beta - k + 1 = 0$ when $2(\beta-k)+1 = -1$). This in turn means that $\omega \leq \pi/(k-1)$. For $k=2$ all angles thus have to be $\leq \pi$, i.e., Ω has to be convex. For $k=3$ all angles have to be $\leq \pi/2$, which is a serious restriction. We refer to Problem 3.10 for an example that illustrates this.

3.8 Problems

Problem 3.1. Give a variational formulation and prove the existence and uniqueness of a weak solution of the Dirichlet problem

$$-\sum_{j,k=1}^{d} \frac{\partial}{\partial x_j}\Big(a_{jk}\frac{\partial u}{\partial x_k}\Big) + a_0 u = f \quad \text{in } \Omega, \quad \text{with } u = 0 \quad \text{on } \Gamma,$$

where $a_{jk}(x)$ and $a_0(x)$ are functions in $\mathcal{C}(\bar{\Omega})$ such that $a_0(x) \geq 0$ and the matrix $(a_{jk}(x))$ is symmetric and uniformly positive definite in Ω, so that $a_{jk}(x) = a_{kj}(x)$ and

$$\sum_{j,k=1}^{d} a_{jk}(x)\xi_j\xi_k \geq \kappa \sum_{j=1}^{d} \xi_j^2 \quad \text{with } \kappa > 0, \text{ for } \xi \in \mathbf{R}^d,\ x \in \Omega.$$

Problem 3.2. Show that if u satisfies $-\Delta u = f$ in $\Omega, u = 0$ on Γ, where $f \in L_2$, then $p = \nabla u$ is the solution to the minimization problem

$$\inf_{q \in H_f} \tfrac{1}{2} \int_\Omega |q|^2 \, \mathrm{d}x,$$

where

$$H_f = \big\{ q = (q_1,\ldots,q_d) : q_i \in L_2,\ -\nabla \cdot q = f \text{ in } \Omega \big\}.$$

Problem 3.3. Consider two bounded domains Ω_1 and Ω_2 with a common boundary S and let $\Gamma_i = \partial\Omega_i \setminus S$, where $\partial\Omega_i$ is the boundary of Ω_i, $i = 1,2$, see Fig. 3.1.

Give a variational formulation of the following problem: Find u_i defined in Ω_i, $i = 1,2$, such that

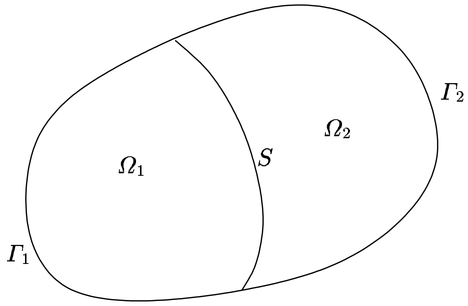

Fig. 3.1. Domain with interface.

$$\begin{aligned} -a_1 \Delta u_1 &= f_1 \quad \text{in } \Omega_1, & -a_2 \Delta u_2 &= f_2 \quad \text{in } \Omega_2, \\ u_1 &= 0 \quad \text{on } \Gamma_1, & u_2 &= 0 \quad \text{on } \Gamma_2, \end{aligned}$$

and

$$u_1 = u_2, \quad a_1 \frac{\partial u_1}{\partial n} = a_2 \frac{\partial u_2}{\partial n} \quad \text{on } S,$$

where $f_i \in L_2(\Omega_i)$, $a_i > 0$ is a constant, for $i = 1, 2$, and n is a unit normal to S. Prove existence and uniqueness of a solution. Give an interpretation from physics.

Problem 3.4. Prove *Friedrichs' inequality*

$$\|v\|_{L_2(\Omega)} \leq C\Big(\|\nabla v\|^2_{L_2(\Omega)} + \|v\|^2_{L_2(\Gamma)}\Big)^{\frac{1}{2}}, \quad \text{for } v \in \mathcal{C}^1,$$

where Ω is a bounded domain in $\mathbf{R}^d$ with boundary Γ. Hint: Integrate by parts in the identity $\int_\Omega v^2 \,\mathrm{d}x = \int_\Omega v^2 \Delta\phi \,\mathrm{d}x$, where $\phi(x) = \frac{1}{2d}|x|^2$.

Problem 3.5. Prove

$$\|v\| \leq C\Big(\|\nabla v\|^2 + \Big(\int_\Omega v \,\mathrm{d}x\Big)^2\Big)^{\frac{1}{2}}, \quad \text{for } v \in \mathcal{C}^1,$$

where Ω is the unit square in $\mathbf{R}^2$. The inequality holds also when Ω is a bounded domain in $\mathbf{R}^d$. Hint: $v(x) = v(y) + \int_{y_1}^{x_1} D_1 v(s, x_2) \,\mathrm{d}s + \int_{y_2}^{x_2} D_2 v(y_1, s) \,\mathrm{d}s$.

Problem 3.6. Give a variational formulation of the problem

$$-\Delta u = f \quad \text{in } \Omega, \quad \text{with } \frac{\partial u}{\partial n} + u = g \quad \text{on } \Gamma,$$

where $f \in L_2(\Omega)$ and $g \in L_2(\Gamma)$. Prove existence and uniqueness of a weak solution. Give an interpretation of the boundary condition in connection with some problem in mechanics or physics. Hint: See Problem 3.4.

Problem 3.7. Prove the stability estimate

$$\|u\|_{H^1(\Omega)} \leq C\Big(\|f\|_{L_2(\Omega)} + \|g\|_{L_2(\Gamma)}\Big)$$

for the solution of (3.34).

Problem 3.8. Give a variational formulation of the problem

$$-\nabla \cdot \big(a\nabla u\big) + cu = f \quad \text{in } \Omega, \quad \text{with } a\frac{\partial u}{\partial n} + h(u-g) = k \quad \text{on } \Gamma,$$

where $f \in L_2(\Omega)$, $g, k \in L_2(\Gamma)$, and the coefficients a, c, h are smooth and such that

$$a(x) \geq a_0 > 0, \ c(x) \geq 0 \quad \text{for } x \in \Omega, \quad h(x) \geq h_0 > 0 \quad \text{for } x \in \Gamma.$$

Prove existence and uniqueness of a weak solution. Prove the stability estimate

$$\|u\|_{H^1(\Omega)} \leq C\big(\|f\|_{L_2(\Omega)} + \|k\|_{L_2(\Gamma)} + \|g\|_{L_2(\Gamma)}\big).$$

Hint: Use Problem 3.4.

Problem 3.9. Consider the Neumann problem

$$-\Delta u = f \quad \text{in } \Omega, \quad \text{with } \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma. \tag{3.38}$$

(a) Assume that $f \in L_2(\Omega)$ and show that the condition

$$\int_\Omega f \,\mathrm{d}x = 0.$$

is necessary for the existence of a solution.

(b) Notice that if u satisfies (3.38), then so does $u + c$ for any constant c. To obtain uniqueness, we add the extra condition

$$\int_\Omega u \,\mathrm{d}x = 0,$$

requiring the mean value of u to be zero. Give this problem a variational formulation using the space

$$V = \Big\{v \in H^1(\Omega) : \int_\Omega v \,\mathrm{d}x = 0\Big\}.$$

Prove that there is a unique weak solution. Hint: See Problem 3.5.

(c) Show that if the weak solution $u \in V$ belongs to H^2, then it solves

$$-\Delta u = f - \int_\Omega f \,\mathrm{d}x \quad \text{in } \Omega, \quad \text{with } \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma.$$

Problem 3.10. Let Ω be a sector with angle $\omega = \pi/\beta$:

$$\Omega = \{(r, \varphi) : 0 < r < 1,\ 0 < \varphi < \pi/\beta\},$$

where r, φ are polar coordinates in the plane. Let $v(r, \varphi) = r^\beta \sin(\beta\varphi)$. Verify that v is harmonic, i.e., $\Delta v = 0$, by computing

$$\Delta v = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial v}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \varphi^2}.$$

(This also follows immediately by noting that v is the imaginary part of the complex analytic function z^β.) Set $u(r, \varphi) = (1 - r^2)v(r, \varphi)$. Then $u = 0$ on Γ. Show that u satisfies $-\Delta u = f$ with $f = 4(1 + \beta)v$. Hence $f \in H^1(\Omega)$. Then compute $\|\partial^2 u/\partial r^2\|_{L_2(\Omega)}$ and conclude that $u \notin H^2(\Omega)$ if $\beta < 1$, i.e., if Ω is non-convex or $\omega > \pi$. Show in a similar way that $u \notin H^3(\Omega)$ if $\omega > \pi/2$. Hint: The most singular term in u_{rr} is $\beta(\beta - 1)r^{\beta-2}\sin(\beta\varphi)$.

Problem 3.11. (Elliptic regularity for a rectangle.) Assume that $\Omega \subset \mathbf{R}^2$ is a rectangle and that u is a smooth function with $u = 0$ on Γ. Prove that

$$|u|_2 = \|\Delta u\|.$$

Use this to prove (3.36) for $\mathcal{A} = -\Delta$.

Hint: Recall that

$$|u|_2^2 = \int_\Omega \Big(\big(\frac{\partial^2 u}{\partial x_1^2}\big)^2 + 2\big(\frac{\partial^2 u}{\partial x_1 \partial x_2}\big)^2 + \big(\frac{\partial^2 u}{\partial x_2^2}\big)^2\Big)\,\mathrm{d}x$$

and integrate by parts in $\int_\Omega \big(\frac{\partial^2 u}{\partial x_1 \partial x_2}\big)^2\,\mathrm{d}x$. Then recall the definition $\|u\|_2 = \big(\|u\|^2 + |u|_1^2 + |u|_2^2\big)^{1/2}$ and prove that $\|u\| \le C|u|_1$ and $|u|_1 \le \big(\|u\|\,|u|_2\big)^{1/2}$.

For arbitrary convex domains one can prove $|u|_2 \le \|\Delta u\|$ by a slightly more complicated argument based on the same idea.

Problem 3.12. Replace the boundary condition in Problem 3.11 by the Neumann condition $\partial u/\partial n = 0$ on Γ. Prove that $|u|_2 = \|\Delta u\|$.

Problem 3.13. (Stability with respect to the coefficient.) Let u_i, $i = 1, 2$, be the weak solutions of the problems

$$-\nabla \cdot \big(a_i \nabla u_i\big) = f \quad \text{in } \Omega, \quad \text{with } u_i = 0 \quad \text{on } \Gamma,$$

where $\Omega \subset \mathbf{R}^d$ is a domain with appropriately smooth boundary Γ, $f \in L_2(\Omega)$, and the coefficients $a_i(x)$ are smooth and such that

$$a_i(x) \ge a_0 > 0 \quad \text{for } x \in \Omega.$$

Prove the stability estimate

$$|u_1 - u_2|_1 \le \frac{C}{a_0^2}\|a_1 - a_2\|_{\mathcal{C}}\|f\|.$$

A Some Tools from Mathematical Analysis

In this appendix we give a short survey of results, essentially without proofs, from mathematical, particularly functional, analysis which are needed in our treatment of partial differential equations. We begin in Sect. A.1 with a simple account of abstract linear spaces with emphasis on Hilbert space, including the Riesz representation theorem and its generalization to bilinear forms of Lax and Milgram. We continue in Sect. A.2 with function spaces, where after a discussion of the spaces $\mathcal{C}^k$, integrability, and the L_p-spaces, we turn to L_2-based Sobolev spaces, with the trace theorem and Poincaré's inequality. The final Sect. A.3 is concerned with the Fourier transform.

A.1 Abstract Linear Spaces

Let V be a linear space (or vector space) with real scalars, i.e., a set such that if $u, v \in V$ and $\lambda, \mu \in \mathbf{R}$, then $\lambda u + \mu v \in V$. A *linear functional* (or *linear form*) L on V is a function $L : V \to \mathbf{R}$ such that

$$L(\lambda u + \mu v) = \lambda L(u) + \mu L(v), \quad \forall u, v \in V, \ \lambda, \mu \in \mathbf{R}.$$

A *bilinear form* $a(\cdot, \cdot)$ on V is a function $a : V \times V \to \mathbf{R}$, which is linear in each argument separately, i.e., such that, for all $u, v, w \in V$ and $\lambda, \mu \in \mathbf{R}$,

$$\begin{aligned} a(\lambda u + \mu v, w) &= \lambda a(u, w) + \mu a(v, w), \\ a(w, \lambda u + \mu v) &= \lambda a(w, u) + \mu a(w, v). \end{aligned}$$

The bilinear form $a(\cdot, \cdot)$ is said to be *symmetric* if

$$a(w, v) = a(v, w), \quad \forall v, w \in V,$$

and *positive definite* if

$$a(v, v) > 0, \quad \forall v \in V, \ v \neq 0.$$

A positive definite, symmetric, bilinear form on V is also called an *inner product* (or *scalar product*) on V. A linear space V with an inner product is called an *inner product space*.

If V is an inner product space and $(\cdot,\cdot)$ is an inner product on V, then we define the corresponding *norm* by

$$\|v\| = (v,v)^{1/2}, \quad \text{for } v \in V. \tag{A.1}$$

We recall the *Cauchy-Schwarz inequality*,

$$|(w,v)| \le \|w\|\,\|v\|, \quad \forall v,w \in V, \tag{A.2}$$

with equality if and only if $w = \lambda v$ or $v = \lambda w$ for some $\lambda \in \mathbf{R}$, and the *triangle inequality*,

$$\|w+v\| \le \|w\| + \|v\|, \quad \forall v,w \in V. \tag{A.3}$$

Two elements $v,w \in V$ for which $(v,w) = 0$ are said to be *orthogonal*.

An infinite sequence $\{v_i\}_{i=1}^{\infty}$ in V is said to converge to $v \in V$, also written $v_i \to v$ as $i \to \infty$ or $v = \lim_{i\to\infty} v_i$, if

$$\|v_i - v\| \to 0 \quad \text{as } i \to \infty.$$

The sequence $\{v_i\}_{i=1}^{\infty}$ is called a *Cauchy sequence* in V if

$$\|v_i - v_j\| \to 0 \quad \text{as } i,j \to \infty.$$

The inner product space V is said to be *complete* if every Cauchy sequence in V is convergent, i.e., if every Cauchy sequence $\{v_i\}_{i=1}^{\infty}$ has a limit $v = \lim v_i \in V$. A complete inner product space is called a *Hilbert space*.

When we want to emphasize that an inner product or a norm is associated to a specific space V, we write $(\cdot,\cdot)_V$ and $\|\cdot\|_V$.

It is sometimes important to permit the scalars in a linear space V to be complex numbers. Such a space is then an inner product space if there is a functional (v,w) defined on $V \times V$, which is linear in the first variable and hermitian, i.e., $(w,v) = \overline{(v,w)}$. The norm is then again defined by (A.1) and V is a complex Hilbert space if completeness holds with respect to this norm. For brevity we generally consider the case of real-valued scalars in the sequel.

More generally, a *norm* in a linear space V is a function $\|\cdot\| : V \to \mathbf{R}_+$ such that

$$\begin{aligned} \|v\| &> 0, && \forall v \in V,\ v \ne 0,\\ \|\lambda v\| &= |\lambda|\,\|v\|, && \forall \lambda \in \mathbf{R} \text{ (or } \mathbf{C}),\ v \in V,\\ \|v+w\| &\le \|v\| + \|w\|, && \forall v,w \in V. \end{aligned}$$

A function $|\cdot|$ is called a *seminorm* if these conditions hold with the exception that the first one is replaced by $|v| \ge 0$, $\forall v \in V$, i.e., if it is only positive semidefinite, and thus can vanish for some $v \ne 0$. A linear space with a norm is called a *normed linear space*. As we have seen, an inner product space is a normed linear space, but not all normed linear spaces are inner product spaces. A complete normed space is called a *Banach space*.

Let V be a Hilbert space and let $V_0 \subset V$ be a linear subspace. Such a subspace V_0 is said to be *closed* if it contains all limits of sequences in V_0, i.e., if $\{v_j\}_{j=1}^{\infty} \subset V_0$ and $v_j \to v$ as $j \to \infty$ implies $v \in V_0$. Such a V_0 is itself a Hilbert space, with the same inner product as V.

Let V_0 be a closed subspace of V. Then any $v \in V$ may be written uniquely as $v = v_0 + w$, where $v_0 \in V_0$ and w is orthogonal to V_0. The element v_0 may be characterized as the unique element in V_0 which is closest to v, i.e.,

$$\|v - v_0\| = \min_{u \in V_0} \|v - u\|. \tag{A.4}$$

This is called the *projection theorem* and is a basic result in Hilbert space theory. The element v_0 is called the *orthogonal projection* of v onto V_0 and is also denoted $P_{V_0}v$. One useful consequence of the projection theorem is that if the closed linear subspace V_0 is not equal to the whole space V, then it has a normal vector, i.e., there is a nonzero vector $w \in V$ which is orthogonal to V_0.

Two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ are said to be *equivalent* in V if there are positive constants c and C such that

$$c\|v\|_b \le \|v\|_a \le C\|v\|_b, \quad \forall v \in V. \tag{A.5}$$

Let V, W be two Hilbert spaces. A linear operator $B : V \to W$ is said to be *bounded*, if there is a constant C such that

$$\|Bv\|_W \le C\|v\|_V, \quad \forall v \in V. \tag{A.6}$$

The norm of a bounded linear operator B is

$$\|B\| = \sup_{v \in V \setminus \{0\}} \frac{\|Bv\|_W}{\|v\|_V}. \tag{A.7}$$

Thus

$$\|Bv\|_W \le \|B\|\,\|v\|_V, \quad \forall v \in V,$$

and, by definition, $\|B\|$ is the smallest constant C such that (A.6) holds.

Note that a bounded linear operator $B : V \to W$ is continuous. In fact, if $v_j \to v$ in V, then $Bv_j \to Bv$ in W as $j \to \infty$, because

$$\|Bv_j - Bv\|_W = \|B(v_j - v)\|_W \le \|B\|\,\|v_j - v\| \to 0, \quad \text{as } j \to \infty.$$

One can show that, conversely, a continuous linear operator is bounded.

In the special case that $W = \mathbf{R}$ the definition of an operator reduces to that of a linear functional. The set of all bounded linear functionals on V is called the *dual space* of V, denoted V^*. By (A.7) the norm in V^* is

$$\|L\|_{V^*} = \sup_{v \in V \setminus \{0\}} \frac{|L(v)|}{\|v\|_V}. \tag{A.8}$$

Note that V^* is itself a linear space if we define $(\lambda L + \mu M)(v) = \lambda L(v) + \mu M(v)$ for $L, M \in V^*$, $\lambda, \mu \in \mathbf{R}$. With the norm defined by (A.8), V^* is a normed linear space, and one may show that V^* is complete, and thus itself also a Banach space.

Similarly, we say that the bilinear form $a(\cdot,\cdot)$ on V is *bounded* if there is a constant M such that

$$|a(w,v)| \leq M\|w\|\,\|v\|, \quad \forall w, v \in V. \tag{A.9}$$

The next theorem states an important property of Hilbert spaces.

Theorem A.1. (Riesz' representation theorem.) *Let V be a Hilbert space with scalar product $(\cdot,\cdot)$. For each bounded linear functional L on V there is a unique $u \in V$ such that*

$$L(v) = (v,u), \quad \forall v \in V.$$

Moreover,

$$\|L\|_{V^*} = \|u\|_V. \tag{A.10}$$

Proof. The uniqueness is clear since $(v,u_1) = (v,u_2)$ with $v = u_1 - u_2$ implies $\|u_1 - u_2\|^2 = (u_1 - u_2, u_1 - u_2) = 0$. If $L(v) = 0$ for all $v \in V$, then we may take $u = 0$. Assume now that $L(\bar{v}) \neq 0$ for some $\bar{v} \in V$. We will construct u as a suitably normalized "normal vector" to the "hyperplane" $V_0 = \{v \in V : L(v) = 0\}$, which is easily seen to be a closed subspace of V, see Problem A.2. Then $\bar{v} = v_0 + w$ with $v_0 \in V_0$ and w orthogonal to V_0 and $L(w) = L(\bar{v}) \neq 0$. But then $L(v - w\,L(v)/L(w)) = 0$, so that $(v - w\,L(v)/L(w), w) = 0$ and hence $L(v) = (v,u)$, $\forall v \in V$, where $u = w\,L(w)/\|w\|^2$. □

This result makes it natural to identify the linear functionals $L \in V^*$ with the associated $u \in V$, and thus V^* is equivalent to V, in the case of a Hilbert space.

We sometimes want to solve equations of the form: Find $u \in V$ such that

$$a(u,v) = L(v), \quad \forall v \in V, \tag{A.11}$$

where V is a Hilbert space, L is a bounded linear functional on V, and $a(\cdot,\cdot)$ is a symmetric bilinear form, which is *coercive* in V, i.e.,

$$a(v,v) \geq \alpha\|v\|_V^2, \quad \forall v \in V, \quad \text{with } \alpha > 0. \tag{A.12}$$

This implies that $a(\cdot,\cdot)$ is symmetric, positive definite, i.e., an inner product on V, and the Riesz representation theorem immediately gives the existence of a unique solution $u \in V$ for each $L \in V^*$.

Moreover, by taking $v = u$ in (A.11) we get

$$\alpha\|u\|_V^2 \leq a(u,u) = L(u) \leq \|L\|_{V^*}\,\|u\|_V,$$

so that, after cancelling one factor $\|u\|_V$,

$$\|u\|_V \le C\|L\|_{V^*}, \quad \text{where } C = 1/\alpha. \tag{A.13}$$

This is an example of an *energy estimate.*

If $a(\cdot,\cdot)$ is a symmetric bilinear form, which is coercive and bounded in V, so that (A.12) and (A.9) hold, then we may define a norm $\|\cdot\|_a$, the *energy norm*, by

$$\|v\|_a = a(v,v)^{1/2}, \quad \text{for } v \in V,$$

By (A.12) and (A.9) we then have

$$\sqrt{\alpha}\|v\|_V \le \|v\|_a \le \sqrt{M}\|v\|_V, \quad \forall v \in V, \tag{A.14}$$

and thus the norm $\|\cdot\|_a$ on V is equivalent to $\|\cdot\|_V$. Clearly, V is then also a Hilbert space with respect to the scalar product $a(\cdot,\cdot)$ and norm $\|\cdot\|_a$.

The solution of (A.11) may also be characterized in terms of a minimization problem.

Theorem A.2. *Assume that $a(\cdot,\cdot)$ is a symmetric, positive definite bilinear form and that L is a bounded linear form on the Hilbert space V. Then $u \in V$ satisfies* (A.11) *if and only if*

$$F(u) \le F(v), \quad \forall v \in V, \quad \textit{where } F(v) = \tfrac{1}{2}a(v,v) - L(v). \tag{A.15}$$

Proof. Suppose first that u satisfies (A.11). Let $v \in V$ be arbitrary and define $w = v - u \in V$. Then $v = u + w$ and

$$\begin{aligned} F(v) &= \tfrac{1}{2}a(u+w,u+w) - L(u+w) \\ &= \tfrac{1}{2}a(u,u) - L(u) + a(u,w) - L(w) + \tfrac{1}{2}a(w,w) \\ &= F(u) + \tfrac{1}{2}a(w,w), \end{aligned}$$

where we have used (A.11) and the symmetry of $a(\cdot,\cdot)$. Since a is positive definite, this proves (A.15).

Conversely, if (A.15) holds, then for $v \in V$ given we have

$$g(t) := F(u+tv) \ge F(u) = g(0), \quad \forall\, t \in \mathbf{R},$$

so that $g(t)$ has a minimum at $t = 0$. But $g(t)$ is the quadratic polynomial

$$\begin{aligned} g(t) &= \tfrac{1}{2}a(u+tv,u+tv) - L(u+tv) \\ &= \tfrac{1}{2}a(u,u) - L(u) + t\big(a(u,v) - L(v)\big) + \tfrac{1}{2}t^2 a(v,v), \end{aligned}$$

and thus $0 = g'(0) = a(u,v) - L(v)$, which is (A.11). □

Thus, $u \in V$ satisfies (A.11) if and only if u minimizes the energy functional F. This method of studying the minimization problem by varying the argument of the functional F around the given vector u is called a variational method, and the equation (A.11) is called the *variational equation* of F.

The following theorem, which is known as the *Lax-Milgram lemma*, extends the Riesz representation theorem to nonsymmetric bilinear forms.

Theorem A.3. *If the bilinear form $a(\cdot,\cdot)$ is bounded and coercive in the Hilbert space V, and L is a bounded linear form in V, then there exists a unique vector $u \in V$ such that* (A.11) *is satisfied. Moreover, the energy estimate* (A.13) *holds.*

Proof. With $(\cdot,\cdot)$ the inner product in V we have by Riesz' representation theorem that there exists a unique $b \in V$ such that

$$L(v) = (b, v), \quad \forall v \in V.$$

Moreover, for each $u \in V$, $a(u,\cdot)$ is clearly also a bounded linear functional on V, so that there exists a unique $A(u) \in V$ such that

$$a(u, v) = (A(u), v), \quad \forall v \in V.$$

It is easy to check that $A(u)$ depends linearly and boundedly on u, so that $Au = A(u)$ defines $A : V \to V$ as a bounded linear operator. The equation (A.11) is therefore equivalent to $Au = b$, and to complete the proof of the theorem we shall show that this equation has a unique solution $u = A^{-1}b$ for each b.

Using the coercivity we have

$$\alpha\|v\|_V^2 \le a(v, v) = (Av, v) \le \|Av\|_V\|v\|_V,$$

so that

$$\alpha\|v\|_V \le \|Av\|_V, \quad \forall v \in V. \tag{A.16}$$

This shows uniqueness, since $Av = 0$ implies $v = 0$. This may also be expressed by saying that the null space $N(A) = \{v \in V : Av = 0\} = 0$, or that A is *injective*.

To show that there exists a solution u for each $b \in V$ means to show that each $b \in V$ belongs to the range $R(A) = \{w \in V : w = Av \text{ for some } v \in V\}$, i.e., $R(A) = V$, or A is *surjective*. To see this we first note that $R(A)$ is a closed linear subspace of V. To show that $R(A)$ is closed, assume that $Av_j \to w$ in V as $j \to \infty$. Then by (A.16) we have $\|v_j - v_i\|_V \le \alpha^{-1}\|Av_j - Av_i\|_V \to 0$ as $i, j \to \infty$. Hence $v_j \to v \in V$ as $j \to \infty$, and by the continuity of A, also $Av_j \to Av = w$. Therefore, $w \in R(A)$ and $R(A)$ is closed.

Assume now that $R(A) \ne V$. Then, by the projection theorem, there exists $w \ne 0$, which is orthogonal to $R(A)$. But, by the orthogonality,

$$\alpha\|w\|_V^2 \le a(w, w) = (Aw, w) = 0,$$

so that $w = 0$, which is a contradiction. Hence $R(A) = V$. This completes the proof that there is a unique solution for each $b \in V$. The energy estimate is proved in the same way as before. □

In the unsymmetric case there is no characterization of the solution in terms of energy minimization.

We finally make a remark about linear equations in finite-dimensional spaces. Let $V = \mathbf{R}^N$ and consider a linear equation in V, which may be written in matrix form as

$$Au = b,$$

where A is a $N \times N$ matrix and u, b are N-vectors. It is well-known that this equation has a unique solution $u = A^{-1}b$ for each $b \in V$, if the matrix A is nonsingular, i.e., if its determinant $\det(A) \neq 0$. If $\det(A) = 0$, then the homogeneous equation $Au = 0$ has nontrivial solutions $u \neq 0$, and $R(A) \neq V$ so that the inhomogeneous equation is not always solvable. Thus we have neither uniqueness nor existence for all $b \in V$. In particular, uniqueness only holds when $\det(A) \neq 0$, and we then also have existence. It is sometimes easy to prove uniqueness, and we then also obtain the existence of the solution at the same time.

A.2 Function Spaces

The Spaces $\mathcal{C}^k$

For $M \subset \mathbf{R}^d$ we denote by $\mathcal{C}(M)$ the linear space of continuous functions on M. The subspace $\mathcal{C}_\mathrm{b}(M)$ of all bounded functions is made into a normed linear space by setting (with a slight abuse of notation)

$$\|v\|_{\mathcal{C}(M)} = \sup_{x \in M} |v(x)|. \tag{A.17}$$

For example, this defines $\|v\|_{\mathcal{C}(\mathbf{R}^d)}$, which we use frequently. When M is a bounded and closed set, i.e., a compact set, the supremum in (A.17) is attained in M and we may write

$$\|v\|_{\mathcal{C}(M)} = \max_{x \in M} |v(x)|.$$

The norm (A.17) is therefore called the *maximum-norm.* Note that convergence in $\mathcal{C}(M)$,

$$\|v_i - v\|_{\mathcal{C}(M)} = \sup_{x \in M} |v_i(x) - v(x)| \to 0, \quad \text{as } i \to \infty,$$

is the same as uniform convergence in M. Recall that if a sequence of continuous functions is uniformly convergent in M, then the limit function is continuous. Using this fact it is not difficult to prove that $\mathcal{C}(M)$ is a complete normed space, i.e., a Banach space. $\mathcal{C}(M)$ is not a Hilbert space, because the maximum-norm is not associated with a scalar product as in (A.1).

Let now $\Omega \subset \mathbf{R}^d$ be a *domain*, i.e., a connected open set. For any integer $k \geq 0$, we denote by $\mathcal{C}^k(\Omega)$ the linear space of all functions v that are k times continuously differentiable in Ω, and by $\mathcal{C}^k(\bar{\Omega})$ the functions in $\mathcal{C}^k(\Omega)$, for which $D^\alpha v \in \mathcal{C}(\bar{\Omega})$ for all $|\alpha| \leq k$, where $D^\alpha v$ denotes the partial derivative of v defined in (1.8). If Ω is bounded, then the latter space is a Banach space with respect to the norm

$$\|v\|_{\mathcal{C}^k(\bar{\Omega})} = \max_{|\alpha| \leq k} \|D^\alpha v\|_{\mathcal{C}(\bar{\Omega})}.$$

For functions in $\mathcal{C}^k(\bar{\Omega})$, $k \geq 1$, we sometimes also use the seminorm containing only the derivatives of highest order,

$$|v|_{\mathcal{C}^k(\bar{\Omega})} = \max_{|\alpha| = k} \|D^\alpha v\|_{\mathcal{C}(\bar{\Omega})}.$$

A function has compact support in Ω if it vanishes outside some compact subset of Ω. We write $\mathcal{C}_0^k(\Omega)$ for the space of functions in $\mathcal{C}^k(\Omega)$ with compact support in Ω. In particular, such functions vanish near the boundary Γ, and for very large x if Ω is unbounded.

We say that a function is *smooth* if, depending on the context, it has sufficiently many continuous derivatives for the purpose at hand.

When there is no risk for confusion, we omit the domain of the functions from the notation of the spaces and write, e.g., $\mathcal{C}$ for $\mathcal{C}(\bar{\Omega})$ and $\|\cdot\|_{\mathcal{C}^k}$ for $\|\cdot\|_{\mathcal{C}^k(\bar{\Omega})}$, and similarly for other spaces that we introduce below.

Integrability, the Spaces L_p

Let Ω be a domain in $\mathbf{R}^d$. We shall need to work with integrals of functions $v = v(x)$ in Ω which are more general than those in $\mathcal{C}(\bar{\Omega})$. For a nonnegative function one may define the so-called *Lebesgue integral*

$$I_\Omega(v) = \int_\Omega v(x)\,\mathrm{d}x,$$

which may be either finite or infinite, and which agrees with the standard Riemann integral for $v \in \mathcal{C}(\bar{\Omega})$. The functions we consider are assumed measurable; we shall not go into details about this concept but just note that all functions that we encounter in this text will satisfy this requirement. A nonnegative function v is said to be integrable if $I_\Omega(v) < \infty$, and a general real or complex-valued function v is similarly integrable if $|v|$ is integrable. A subset Ω_0 of Ω is said to be a nullset, or a set of measure 0, if its volume $|\Omega|$ equals 0. Two functions which are equal except on a nullset are said to be equal almost everywhere (a.e.), and they then have the same integral. Thus if $v_1(x) = 1$ in a bounded domain Ω and if $v_2(x) = 1$ in Ω except at $x_0 \in \Omega$ where $v_2(x_0) = 2$, then $I_\Omega(v_1) = I_\Omega(v_2) = |\Omega|$. In particular, from the fact that a function is integrable we cannot draw any conclusion about its value at a point $x_0 \in \Omega$, i.e., the point values are not well defined. Also, since the boundary Γ of Ω is a nullset, $I_{\bar{\Omega}}(v) = I_{\Omega}(v)$ for any v.

We now define

$$\|v\|_{L_p} = \|v\|_{L_p(\Omega)} = \begin{cases} \Big(\int_\Omega |v(x)|^p \, \mathrm{d}x\Big)^{1/p}, & \text{for } 1 \le p < \infty, \\ \operatorname*{ess\,sup}_\Omega |v(x)|, & \text{for } p = \infty, \end{cases}$$

and say that $v \in L_p = L_p(\Omega)$ if $\|v\|_{L_p} < \infty$. Here the ess sup means the *essential supremum*, disregarding values on nullsets, so that, e.g., $\|v_2\|_{L_\infty} = 1$ for the function v_2 above, even though $\sup_\Omega v_2 = 2$. One may show that L_p is a complete normed space, i.e., a Banach space; the triangle inequality in L_p is called Minkowski's inequality. Clearly, any $v \in \mathcal{C}$ belongs to L_p for $1 \le p \le \infty$ if Ω is bounded, and

$$\|v\|_{L_p} \le C\|v\|_{\mathcal{C}}, \quad \text{with } C = |\Omega|^{1/p}, \quad \text{for } 1 \le p < \infty, \quad \text{and } \|v\|_{L_\infty} = \|v\|_{\mathcal{C}},$$

but L_p also contains functions that are not continuous. Moreover, it is not difficult to show that $\mathcal{C}(\bar{\Omega})$ is incomplete with respect to the L_p-norm for $1 \le p < \infty$. To see this one constructs a sequence $\{v_i\}_{i=1}^\infty \subset \mathcal{C}(\bar{\Omega})$, which is a Cauchy sequence with respect to the L_p-norm, i.e., such that $\|v_i - v_j\|_{L_p} \to 0$, but whose limit $v = \lim_{i\to\infty} v_i$ is discontinuous. However, $\mathcal{C}(\bar{\Omega})$ is a *dense subspace* of $L_p(\Omega)$ for $1 \le p < \infty$, if Γ is sufficiently smooth. By this we mean that for any $v \in L_p$ there is a sequence $\{v_i\}_{i=1}^\infty \subset \mathcal{C}$ such that $\|v_i - v\|_{L_p} \to 0$ as $i \to \infty$. In other words, any function $v \in L_p$ can be approximated arbitrarily well in the L_p-norm by functions in $\mathcal{C}$ (in fact, for any k by functions in $\mathcal{C}_0^k$). In contrast, $\mathcal{C}$ is not dense in L_∞ since a discontinuous function cannot be well approximated uniformly by a continuous function.

The case L_2 is of particular interest to us, and this space is an inner product space, and hence a Hilbert space, with respect to the inner product

$$(v, w) = \int_\Omega v(x)w(x) \, \mathrm{d}x. \tag{A.18}$$

In the case of complex-valued functions one takes the complex conjugate of $w(x)$ in the integrand.

Sobolev Spaces

We shall now introduce some particular Hilbert spaces which are natural to use in the study of partial differential equations. These spaces consist of functions which are square integrable together with their partial derivatives up to a certain order. To define them we first need to generalize the concept of a partial derivative.

Let Ω be a domain in $\mathbf{R}^d$ and let first $v \in \mathcal{C}^1(\bar{\Omega})$. Integration by parts yields

$$\int_\Omega \frac{\partial v}{\partial x_i}\phi \,\mathrm{d}x = -\int_\Omega v \frac{\partial \phi}{\partial x_i}\,\mathrm{d}x, \quad \forall \phi \in \mathcal{C}_0^1 = \mathcal{C}_0^1(\Omega).$$

If $v \in L_2 = L_2(\Omega)$, then $\partial v/\partial x_i$ does not necessarily exist in the classical sense, but we may define $\partial v/\partial x_i$ to be the linear functional

$$L(\phi) = \frac{\partial v}{\partial x_i}(\phi) = -\int_\Omega v \frac{\partial \phi}{\partial x_i}\,\mathrm{d}x, \quad \forall \phi \in \mathcal{C}_0^1. \tag{A.19}$$

This functional is said to be a *generalized* or *weak derivative* of v. When L is bounded in L_2, i.e., $|L(\phi)| \le C\|\phi\|$, it follows from Riesz' representation theorem that there exists a unique function $w \in L_2$, such that $L(\phi) = (w, \phi)$ for all $\phi \in L_2$, and in particular

$$-\int_\Omega v \frac{\partial \phi}{\partial x_i}\,\mathrm{d}x = \int_\Omega w\phi \,\mathrm{d}x, \quad \forall \phi \in \mathcal{C}_0^1.$$

We then say that the weak derivative belongs to L_2 and write $\partial v/\partial x_i = w$. In this case we thus have

$$\int_\Omega \frac{\partial v}{\partial x_i}\phi \,\mathrm{d}x = -\int_\Omega v \frac{\partial \phi}{\partial x_i}\,\mathrm{d}x, \quad \forall \phi \in \mathcal{C}_0^1. \tag{A.20}$$

In particular, if $v \in \mathcal{C}^1(\bar{\Omega})$, then the generalized derivative $\partial v/\partial x_i$ coincides with the classical derivative $\partial v/\partial x_i$.

In a similar way, with $D^\alpha v$ denoting the partial derivative of v defined in (1.8), we define the weak partial derivative $D^\alpha v$ as the linear functional

$$D^\alpha v(\phi) = (-1)^{|\alpha|}\int_\Omega v\, D^\alpha \phi \,\mathrm{d}x, \quad \forall \phi \in \mathcal{C}_0^{|\alpha|}. \tag{A.21}$$

When this functional is bounded in L_2, Riesz' representation theorem shows that there exists a unique function in L_2, which we denote by $D^\alpha v$, such that

$$(D^\alpha v, \phi) = (-1)^{|\alpha|}(v, D^\alpha \phi), \quad \forall \phi \in \mathcal{C}_0^{|\alpha|}.$$

We refer to Problem A.9 for further discussion of generalized functions.

We now define $H^k = H^k(\Omega)$, for $k \ge 0$, to be the space of all functions whose weak partial derivatives of order $\le k$ belong to L_2, i.e.,

$$H^k = H^k(\Omega) = \big\{v \in L_2 : D^\alpha v \in L_2 \text{ for } |\alpha| \le k\big\},$$

and we equip this space with the inner product

$$(v, w)_k = (v, w)_{H^k} = \sum_{|\alpha|\le k}\int_\Omega D^\alpha v D^\alpha w \,\mathrm{d}x,$$

and the corresponding norm

$$\|v\|_k = \|v\|_{H^k} = (v,v)_{H^k}^{1/2} = \Big(\sum_{|\alpha|\le k} \int_\Omega (D^\alpha v)^2 \, \mathrm{d}x \Big)^{1/2}.$$

In particular, $\|v\|_0 = \|v\|_{L_2}$, and in this case we normally omit the subscript 0 and write $\|v\|$. Also

$$\|v\|_1 = \Big(\int_\Omega \Big\{ v^2 + \sum_{j=1}^d \big(\frac{\partial v}{\partial x_j}\big)^2 \Big\} \, \mathrm{d}x \Big)^{1/2} = \Big(\|v\|^2 + \|\nabla v\|^2 \Big)^{1/2}$$

and

$$\|v\|_2 = \Big(\int_\Omega \Big\{ v^2 + \sum_{j=1}^d \big(\frac{\partial v}{\partial x_j}\big)^2 + \sum_{i,j=1}^d \big(\frac{\partial^2 v}{\partial x_i \partial x_j}\big)^2 \Big\} \, \mathrm{d}x \Big)^{1/2}.$$

We sometimes also use the seminorm, for $k \ge 1$,

$$|v|_k = |v|_{H^k} = \Big(\sum_{|\alpha|=k} \int_\Omega (D^\alpha v)^2 \, \mathrm{d}x \Big)^{1/2}. \tag{A.22}$$

Note that the seminorm vanishes for constant functions. Using the fact that L_2 is complete, one may show that H^k is complete and thus a Hilbert space, see Problem A.4. The space H^k is an example of a more general class of function spaces called Sobolev spaces.

It may be shown that $\mathcal{C}^l = \mathcal{C}^l(\bar\Omega)$ is dense in $H^k = H^k(\Omega)$ for any $l \ge k$, if Γ is sufficiently smooth. This is useful because it allows us to obtain certain results for H^k by carrying out the proof for functions in $\mathcal{C}^k$, which may be technically easier, and then extend the result to all $v \in H^k$ by using the density, cf. the proof of Theorem A.4 below.

Similarly, we denote by $W_p^k = W_p^k(\Omega)$ the normed space defined by the norm

$$\|v\|_{W_p^k} = \Big(\int_\Omega \sum_{|\alpha|\le k} |D^\alpha v|^p \, \mathrm{d}x \Big)^{1/p}, \quad \text{for } 1 \le p < \infty,$$

with the obvious modification for $p = \infty$. This space is in fact complete and hence a Banach space. For $p = 2$ we have $W_2^k = H^k$. Again, for $v \in \mathcal{C}^k$ we have $\|v\|_{W_\infty^k} = \|v\|_{\mathcal{C}^k}$.

Trace Theorems

If $v \in \mathcal{C}(\bar\Omega)$, then $v(x)$ is well defined for $x \in \Gamma$, the boundary of Ω. The *trace* γv of such a v on Γ is the restriction of v to Γ, i.e.,

$$(\gamma v)(x) = v(x), \quad \text{for } x \in \Gamma. \tag{A.23}$$

Recall that since Γ is a nullset, the trace of $v \in L_2(\Omega)$ is not well defined.

Suppose now that $v \in H^1(\Omega)$. Is it then possible to define v uniquely on Γ, i.e., to define its trace γv on Γ? (One may show that functions in

$H^1(\Omega)$ are not necessarily continuous, cf. Theorem A.5 and Problems A.6, A.7 below.) This question can be made more precise by asking if it is possible to find a norm $\|\cdot\|_{(\Gamma)}$ for functions on Γ and some constant C that

$$\|\gamma v\|_{(\Gamma)} \leq C\|v\|_1, \quad \forall v \in \mathcal{C}^1(\bar{\Omega}). \tag{A.24}$$

An inequality of this form is called a *trace inequality.* If (A.24) holds, then by a density argument (see below) it is possible to extend the domain of definition of the trace operator γ from $\mathcal{C}^1(\bar{\Omega})$ to $H^1(\Omega)$, and (A.24) will also hold for all $v \in H^1(\Omega)$. The function space to which γv will belong will be defined by the norm $\|\cdot\|_{(\Gamma)}$ in (A.24).

We remark that in the above discussion the boundary Γ could be replaced by some other subset of Ω of dimension smaller than d.

In order to proceed with the trace theorems, we first consider a one-dimensional case, with Γ corresponding to a single point.

Lemma A.1. *Let $\Omega = (0,1)$. Then there is a constant C such that*

$$|v(x)| \leq C\|v\|_1, \quad \forall x \in \bar{\Omega},\ \forall v \in \mathcal{C}^1(\bar{\Omega}).$$

Proof. For $x, y \in \Omega$ we have $v(x) = v(y) + \int_y^x v'(s)\,\mathrm{d}s$, and hence by the Cauchy-Schwarz inequality

$$|v(x)| \leq |v(y)| + \int_0^1 |v'(s)|\,\mathrm{d}s \leq |v(y)| + \|v'\|.$$

Squaring both sides and integrating with respect to y, we obtain,

$$v(x)^2 \leq 2\big(\|v\|^2 + \|v'\|^2\big) = 2\|v\|_1^2. \tag{A.25}$$

which shows the desired estimate. □

We now show a simple trace theorem. By $L_2(\Gamma)$ we denote the Hilbert space of all functions that are square integrable on Γ with norm

$$\|w\|_{L_2(\Gamma)} = \Big(\int_\Gamma w^2\,\mathrm{d}s\Big)^{1/2}.$$

Theorem A.4. (Trace theorem.) *Let Ω be a bounded domain in $\mathbf{R}^d$ $(d \geq 2)$ with smooth or polygonal boundary Γ. Then the trace operator $\gamma : \mathcal{C}^1(\bar{\Omega}) \to \mathcal{C}(\Gamma)$ may be extended to $\gamma : H^1(\Omega) \to L_2(\Gamma)$, which defines the trace $\gamma v \in L_2(\Gamma)$ for $v \in H^1(\Omega)$. Moreover, there is a constant $C = C(\Omega)$ such that*

$$\|\gamma v\|_{L_2(\Gamma)} \leq C\|v\|_1, \quad \forall v \in H^1(\Omega). \tag{A.26}$$

Proof. We first show the trace inequality (A.26) for functions $v \in \mathcal{C}^1(\bar{\Omega})$. For simplicity we consider only the case when $\Omega = (0,1) \times (0,1)$, the unit square in $\mathbf{R}^2$. The proof in the general case is similar. For $x = (x_1, x_2) \in \Omega$ we have by (A.25)

$$v(0,x_2)^2 \le 2\Big(\int_0^1 v(x_1,x_2)^2\,\mathrm{d}x_1 + \int_0^1 \Big(\frac{\partial v}{\partial x_1}(x_1,x_2)\Big)^2\,\mathrm{d}x_1\Big),$$

and hence by integration with respect to x_2,

$$\int_0^1 v(0,x_2)^2\,\mathrm{d}x_2 \le 2\big(\|v\|^2 + \|\nabla v\|^2\big) = 2\|v\|_1^2.$$

The analogous estimates for the remaining parts of Γ complete the proof of (A.26) for $v \in \mathcal{C}^1$.

Let now $v \in H^1(\Omega)$. Since $\mathcal{C}^1$ is dense in H^1 there is a sequence $\{v_i\}_{i=1}^\infty \subset \mathcal{C}^1$ such that $\|v - v_i\|_1 \to 0$. This sequence is then a Cauchy sequence in H^1, i.e., $\|v_i - v_j\|_1 \to 0$ as $i, j \to \infty$. Applying (A.26) to $v_i - v_j \in \mathcal{C}^1$, we find

$$\|\gamma v_i - \gamma v_j\|_{L_2(\Gamma)} \le C\|v_i - v_j\|_1 \to 0, \quad \text{as } i, j \to \infty,$$

i.e., $\{\gamma v_i\}_{i=1}^\infty$ is a Cauchy sequence in $L_2(\Gamma)$, and thus there exists $w \in L_2(\Gamma)$ such that $\gamma v_i \to w$ in $L_2(\Gamma)$ as $i \to \infty$. We define $\gamma v = w$. It is easy to show that (A.26) then holds for $v \in H^1$. This extends γ to a bounded linear operator $\gamma : H^1(\Omega) \to L_2(\Gamma)$. Since $\mathcal{C}^1$ is dense in H^1, there is only one such extension (prove this!). In particular, γ is independent of the choice of the sequence $\{v_i\}$. □

The constant in Theorem A.4 depends on the size and shape of the domain Ω. It is sometimes important to have more detailed information about this dependence, and in Problem A.15 we assume that the shape is fixed (a square) and investigate the dependence of the constant on the size of Ω.

The following result, of a somewhat similar nature, is a special case of the well-known and important Sobolev inequality.

Theorem A.5. *Let Ω be a bounded domain in $\mathbf{R}^d$ with smooth or polygonal boundary and let $k > d/2$. Then $H^k(\Omega) \subset \mathcal{C}(\bar{\Omega})$, and there exists a constant $C = C(\Omega)$ such that*

$$\|v\|_{\mathcal{C}} \le C\|v\|_k, \quad \forall v \in H^k(\Omega). \tag{A.27}$$

In the same way as for the trace theorem it suffices to show (A.27) for smooth v, see Problem A.20. The particular case when $d = k = 1$ is given in Lemma A.1, and Problem A.13 considers the case $\Omega = (0,1) \times (0,1)$. The general case is more complicated. As shown in Problems A.6, A.7, a function in $H^1(\Omega)$ with $\Omega \subset \mathbf{R}^d$ is not necessarily continuous when $d \ge 2$.

If we apply Sobolev's inequality to derivatives of v, we get

$$\|v\|_{\mathcal{C}^\ell} \le C\|v\|_k, \quad \forall v \in H^k(\Omega), \text{ if } k > \ell + d/2, \tag{A.28}$$

and we may similarly conclude that $H^k(\Omega) \subset \mathcal{C}^\ell(\bar{\Omega})$ if $k > \ell + d/2$.

The Space $H_0^1(\Omega)$. Poincaré's Inequality

Theorem A.4 shows that the trace operator $\gamma : H^1(\Omega) \to L_2(\Gamma)$ is a bounded linear operator. This implies that its null space,

$$H_0^1(\Omega) = \{v \in H^1(\Omega) : \gamma v = 0\},$$

is a closed subspace of $H^1(\Omega)$, and hence a Hilbert space with the norm $\|\cdot\|_1$. It is the set of functions in H^1 that vanish on Γ in the sense of trace. We note that the seminorm $|v|_1 = \|\nabla v\|$ defined in (A.22) is in fact a norm on $H_0^1(\Omega)$, equivalent to $\|\cdot\|_1$, as follows from the following result.

Theorem A.6. (Poincaré's inequality.) *If Ω is a bounded domain in $\mathbf{R}^d$, then there exists a constant $C = C(\Omega)$ such that*

$$\|v\| \le C\|\nabla v\|, \quad \forall v \in H_0^1(\Omega). \tag{A.29}$$

Proof. As an example we show the result for $\Omega = (0,1) \times (0,1)$. The proof in the general case is similar.

Since $\mathcal{C}_0^1$ is dense in H_0^1, it suffices to show (A.29) for $v \in \mathcal{C}_0^1$. For such a v we write

$$v(x) = \int_0^{x_1} \frac{\partial v}{\partial x_1}(s, x_2)\,\mathrm{d}s, \quad \text{for } x = (x_1, x_2) \in \Omega,$$

and hence by the Cauchy-Schwarz inequality

$$|v(x)|^2 \le \int_0^1 \mathrm{d}s \int_0^1 \Big(\frac{\partial v}{\partial x_1}(s, x_2)\Big)^2\,\mathrm{d}s.$$

The result now follows by integration with respect to x_2 and x_1, with $C = 1$ in this case. $\square$

The equivalence of the norms $|\cdot|_1$ and $\|\cdot\|_1$ on $H_0^1(\Omega)$ now follows from

$$\|\nabla v\|^2 \le \|v\|_1^2 = \|v\|^2 + \|\nabla v\|^2 \le (C+1)\|\nabla v\|^2, \quad \forall v \in H_0^1(\Omega).$$

The dual space of $H_0^1(\Omega)$ is denoted $H^{-1}(\Omega)$. Thus $H^{-1} = (H_0^1)^*$ is the space of all bounded linear functionals on H_0^1. The norm in H^{-1} is (cf. (A.8))

$$\|L\|_{(H_0^1)^*} = \|L\|_{-1} = \sup_{v \in H_0^1} \frac{|L(v)|}{|v|_1}. \tag{A.30}$$

A.3 The Fourier Transform

Let v be a real or complex function in $L_1(\mathbf{R}^d)$. We define its Fourier transform for $\xi = (\xi_1, \ldots, \xi_d) \in \mathbf{R}^d$ by

$$\mathcal{F}v(\xi) = \hat{v}(\xi) = \int_{\mathbf{R}^d} v(x)\mathrm{e}^{-\mathrm{i}x\cdot\xi}\,\mathrm{d}x, \quad \text{where } x\cdot\xi = \sum_{j=1}^{d} x_j\xi_j.$$

The inverse Fourier transform is

$$\mathcal{F}^{-1}v(x) = \check{v}(x) = (2\pi)^{-d}\int_{\mathbf{R}^d} v(\xi)\mathrm{e}^{\mathrm{i}x\cdot\xi}\,\mathrm{d}\xi = (2\pi)^{-d}\hat{v}(-x), \quad \text{for } x \in \mathbf{R}^d.$$

If v and $\hat{v}$ are both in $L_1(\mathbf{R}^d)$, then Fourier's inversion formula holds, i.e.,

$$\mathcal{F}^{-1}(\mathcal{F}\,v) = (\hat{v})\check{\ } = v.$$

The inner product in $L_2(\mathbf{R}^d)$ of two functions can be expressed in terms of their Fourier transforms according to Parseval's formula,

$$\int_{\mathbf{R}^d} v(x)\overline{w(x)}\,\mathrm{d}x = (2\pi)^{-d}\int_{\mathbf{R}^d} \hat{v}(\xi)\overline{\hat{w}(\xi)}\,\mathrm{d}\xi, \tag{A.31}$$

or

$$(v,w) = (2\pi)^{-d}(\hat{v},\hat{w}), \quad \text{where } (v,w) = (v,w)_{L_2(\mathbf{R}^d)}.$$

In particular, we have for the corresponding norms

$$\|v\| = (2\pi)^{-d/2}\|\hat{v}\|. \tag{A.32}$$

Let $D^\alpha v$ be a partial derivative of v as defined in (1.8). We then have, assuming that v and its derivatives are sufficiently small for $|x|$ large,

$$\mathcal{F}(D^\alpha v)(\xi) = (i\xi)^\alpha\hat{v}(\xi) = i^{|\alpha|}\xi^\alpha\hat{v}(\xi), \quad \text{where } \xi^\alpha = \xi_1^{\alpha_1}\cdots\xi_d^{\alpha_d}.$$

In fact, by integration by parts,

$$\int_{\mathbf{R}^d} D^\alpha v(x)\mathrm{e}^{-\mathrm{i}x\cdot\xi}\,\mathrm{d}x = (-1)^{|\alpha|}\int_{\mathbf{R}^d} v(x)D^\alpha(\mathrm{e}^{-\mathrm{i}x\cdot\xi})\,\mathrm{d}x = (i\xi)^\alpha\hat{v}(\xi).$$

Further, translation of the argument of the function corresponds to multiplication of its Fourier transform by an exponential,

$$\mathcal{F}v(\cdot+y)(\xi) = \mathrm{e}^{\mathrm{i}y\cdot\xi}\hat{v}(\xi), \quad \text{for } y \in \mathbf{R}^d, \tag{A.33}$$

and for scaling of the argument we have

$$\mathcal{F}v(a\cdot)(\xi) = a^{-d}\hat{v}(a^{-1}\xi), \quad \text{for } a > 0. \tag{A.34}$$

The convolution of two functions v and w is defined by

$$(v*w)(x) = \int_{\mathbf{R}^d} v(x-y)w(y)\,\mathrm{d}y = \int_{\mathbf{R}^d} v(y)w(x-y)\,\mathrm{d}y,$$

and we have

$$\mathcal{F}(v * w)(\xi) = \hat{v}(\xi)\hat{w}(\xi),$$

because

$$\begin{aligned}
\int_{\mathbf{R}^d} \Big(\int_{\mathbf{R}^d} & v(x-y)w(y)\,\mathrm{d}y \Big) \mathrm{e}^{-\mathrm{i}x\cdot\xi}\,\mathrm{d}x \\
&= \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} v(x-y)w(y)\mathrm{e}^{-\mathrm{i}(x-y)\cdot\xi}\mathrm{e}^{-\mathrm{i}y\cdot\xi}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} v(z)w(y)\mathrm{e}^{-\mathrm{i}z\cdot\xi}\mathrm{e}^{-\mathrm{i}y\cdot\xi}\,\mathrm{d}z\,\mathrm{d}y.
\end{aligned}$$

It follows, which can also easily be shown directly, that differentiation of a convolution can be carried out on either factor,

$$D^\alpha(v * w) = D^\alpha v * w = v * D^\alpha w.$$

A.4 Problems

Problem A.1. Let V be a Hilbert space with scalar product $(\cdot,\cdot)$ and let $u \in V$ be given. Define $L : V \to \mathbf{R}$ by $L(v) = (u, v)\ \forall v \in V$. Prove that L is a bounded linear functional on V. Determine $\|L\|$.

Problem A.2. Prove that if $L : V \to \mathbf{R}$ is a bounded linear functional and $\{v_i\}$ is a sequence with $L(v_i) = 0$ that converges to $v \in V$, then $L(v) = 0$. This proves that the subspace V_0 in the proof of Theorem A.1 is closed.

Problem A.3. Prove the energy estimate (A.13) by using (A.10) and (A.14). Hint: Recall (A.8) and note that (A.10) means

$$\sup_{v\in V\setminus\{0\}} \frac{|L(v)|}{\|v\|_a} = \|u\|_a.$$

Problem A.4. Given that $L_2(\Omega)$ is complete, prove that $H^1(\Omega)$ is complete. Hint: Assume that $\|v_j - v_i\|_1 \to 0$ as $i, j \to \infty$. Show that there are v, w_k such that $\|v_j - v\| \to 0$, $\|\partial v_j/\partial x_k - w_k\| \to 0$, and that $w_k = \partial v/\partial x_k$ in the sense of weak derivative.

Problem A.5. Let $\Omega = (-1, 1)$ and let $v : \Omega \to \mathbf{R}$ be defined by $v(x) = 1$ if $x \in (-1, 0)$ and $v(x) = 0$ if $x \in (0, 1)$. Prove that $v \in L_2(\Omega)$ and that v can be approximated arbitrarily well in L_2-norm by $\mathcal{C}^0$-functions.

Problem A.6. Let Ω be the unit ball in $\mathbf{R}^d$, $d = 1, 2, 3$, i.e., $\Omega = \{x \in \mathbf{R}^d : |x| < 1\}$. For which values of $\lambda \in \mathbf{R}$ does the function $v(x) = |x|^\lambda$ belong to (a) $L_2(\Omega)$, (b) $H^1(\Omega)$?

Problem A.7. Check if the function $v(x) = \log(-\log|x|^2)$ belongs to $H^1(\Omega)$ if $\Omega = \{x \in \mathbf{R}^2 : |x| < \frac{1}{2}\}$. Are functions in $H^1(\Omega)$ necessarily bounded and continuous?

Problem A.8. It is known that $\mathcal{C}_0^1(\Omega)$ is dense in $L_2(\Omega)$ and $H_0^1(\Omega)$. Explain why $\mathcal{C}_0^1(\Omega)$ is not dense in $H^1(\Omega)$.

Problem A.9. The generalized (or weak) derivative defined in (A.19) is a special case of the so-called *generalized functions* or *distributions*. Another important example is *Dirac's delta*, which is defined as a linear functional acting on continuous test functions, for $\Omega \subset \mathbf{R}^d$,

$$\delta(\phi) = \phi(0), \quad \forall \phi \in \mathcal{C}_0^0(\Omega).$$

Let now $d = 1$, $\Omega = (-1, 1)$ and

$$f(x) = \begin{cases} x, & x \geq 0, \\ 0, & x \leq 0, \end{cases} \qquad g(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

Show that $f' = g$, $g' = \delta$ in the sense of generalized derivative, i.e.,

$$\begin{aligned} f'(\phi) &= -\int_\Omega f\phi' \,\mathrm{d}x = \int_\Omega g\phi \,\mathrm{d}x, && \forall \phi \in \mathcal{C}_0^1(\Omega), \\ g'(\phi) &= -\int_\Omega g\phi' \,\mathrm{d}x = \phi(0), && \forall \phi \in \mathcal{C}_0^1(\Omega). \end{aligned}$$

Conclude that the generalized derivative $f' = g$ belongs to L_2, but that $g' = \delta$ does not. For the latter statement, you must show that δ is not bounded with respect to the L_2-norm, i.e., you need to find a sequence of test functions such that $\|\phi_i\|_{L_2} \to 0$, but $\phi_i(0) = 1$ as $i \to \infty$. Thus, $f \in H^1(\Omega)$ and $g \notin H^1(\Omega)$.

Problem A.10. For $f \in L_2(\Omega)$ we define the linear functional $f(v) = (f, v)$ $\forall v \in L_2(\Omega)$. Show the inequality, cf. (A.30),

$$\|f\|_{-1} \leq C\|f\|, \quad \forall f \in L_2(\Omega).$$

Conclude that $L_2(\Omega) \subset H^{-1}(\Omega)$.

Problem A.11. Let $\Omega = (0, 1)$ and $f(x) = 1/x$. Show that $f \notin L_2(\Omega)$. Show that $f \in H^{-1}(\Omega)$ by defining the linear functional $f(v) = (f, v)$ $\forall v \in H_0^1(\Omega)$, and proving the inequality

$$|(f, v)| \leq C\|v'\|, \quad \forall v \in H_0^1(\Omega).$$

Conclude that $H^{-1}(\Omega) \not\subset L_2(\Omega)$.

Problem A.12. Prove that if $\Omega = (0, L)$ is a finite interval, then there is a constant $C = C(L)$ such that, for all $x \in \bar{\Omega}$ and $v \in \mathcal{C}^1(\bar{\Omega})$,

$$\text{(a)} \qquad |v(x)| \leq L^{-1} \int_\Omega |v| \,\mathrm{d}y + \int_\Omega |v'| \,\mathrm{d}y \leq C\|v\|_{W_1^1(\Omega)},$$

$$\text{(b)} \qquad |v(x)|^2 \leq L^{-1} \int_\Omega |v|^2 \,\mathrm{d}y + L \int_\Omega |v'|^2 \,\mathrm{d}y \leq C\|v\|_1^2,$$

$$\text{(c)} \qquad |v(x)|^2 \leq L^{-1}\|v\|^2 + 2\|v\|\,\|v'\| \leq C\|v\|\,\|v\|_1.$$

Problem A.13. Prove that if Ω is the unit square in $\mathbf{R}^2$, then there exists a constant C such that

$$\text{(a)} \qquad \|v\|_{L_1(\Gamma)} \le C\|v\|_{W_1^1(\Omega)}, \qquad \forall v \in \mathcal{C}^1(\bar{\Omega}),$$
$$\text{(b)} \qquad \|v\|_{\mathcal{C}} \le C\|v\|_{W_1^2}, \qquad \forall v \in \mathcal{C}^2(\bar{\Omega}).$$

Since $\|v\|_{W_1^2} \le 3^{1/2}|\Omega|^{1/2}\|v\|_{H^2}$, part (b) implies the special case of Theorem A.5 with $k = d = 2$ and Ω a square domain. Part (b) directly generalizes to $\|v\|_{\mathcal{C}} \le C\|v\|_{W_1^d}$ for $\Omega \subset \mathbf{R}^d$. Hint: Proof of Theorem A.4.

Problem A.14. (Scaling of Sobolev norms.) Let L be a positive number and consider the coordinate transformation $x = L\hat{x}$, which maps the bounded domain $\Omega \subset \mathbf{R}^d$ onto $\hat{\Omega}$. A function v defined on Ω is transformed to a function $\hat{v}$ on $\hat{\Omega}$ according to $\hat{v}(\hat{x}) = v(L\hat{x})$. Prove the scaling identities

$$\text{(a)} \qquad \|v\|_{L_2(\Omega)} = L^{d/2}\|\hat{v}\|_{L_2(\hat{\Omega})},$$
$$\text{(b)} \qquad \|\nabla v\|_{L_2(\Omega)} = L^{d/2-1}\|\hat{\nabla}\hat{v}\|_{L_2(\hat{\Omega})},$$
$$\text{(c)} \qquad \|v\|_{L_2(\Gamma)} = L^{d/2-1/2}\|\hat{v}\|_{L_2(\hat{\Gamma})}.$$

Problem A.15. (Scaled trace inequality.) Let $\Omega = (0, L) \times (0, L)$ be a square domain of side L. Prove the scaled trace inequality

$$\|v\|_{L_2(\Gamma)} \le C\Big(L^{-1}\|v\|^2_{L_2(\Omega)} + L\|\nabla v\|^2_{L_2(\Omega)}\Big)^{1/2}, \quad \forall v \in \mathcal{C}^1(\bar{\Omega}).$$

Hint: Apply (A.26) with $\hat{\Omega} = (0, 1) \times (0, 1)$ and use the scaling identities in Problem A.14.

Problem A.16. Let Ω be the unit square in $\mathbf{R}^2$. Prove the trace inequality in the form

$$\|v\|^2_{L_2(\Gamma)} \le C\big(\|v\|^2_{L_2(\Omega)} + \|v\|_{L_2(\Omega)}\|\nabla v\|_{L_2(\Omega)}\big) \le C\|v\|\,\|v\|_1.$$

Hint: Start from

$$v(0, y_2)^2 = v(y_1, y_2)^2 - \int_0^{y_1} \frac{\partial}{\partial x_1} v(s, y_2)^2 \, \mathrm{d}s.$$

Problem A.17. It is a fact from linear algebra that all norms on a finite-dimensional space V are equivalent. Illustrate this by proving the following norm equivalences in $V = \mathbf{R}^N$:

$$\text{(A.35)} \qquad \|v\|_{l_2} \le \|v\|_{l_1} \le \sqrt{N}\|v\|_{l_2},$$
$$\text{(A.36)} \qquad \|v\|_{l_\infty} \le \|v\|_{l_2} \le \sqrt{N}\|v\|_{l_\infty},$$
$$\text{(A.37)} \qquad \|v\|_{l_\infty} \le \|v\|_{l_1} \le N\|v\|_{l_\infty},$$

where

$$\|v\|_{l_p} = \Big(\sum_{j=1}^{N} |v_j|^p\Big)^{1/p} \quad \text{for } 1 \le p < \infty, \quad \|v\|_{l_\infty} = \max_{1\le j\le N} |v_j|.$$

Note that the equivalence constants tend to infinity as $N \to \infty$.

Problem A.18. Prove (A.33) and (A.34).

Problem A.19. Prove that the Fourier transform of $v(x) = \mathrm{e}^{-|x|^2}$ is $\hat{v}(\xi) = \pi^{d/2}\mathrm{e}^{-|\xi|^2/4}$.

Problem A.20. Assume that Sobolev's inequality in (A.27) has been proved for all $v \in \mathcal{C}^k(\bar{\Omega})$ with $k > d/2$. Prove Sobolev's imbedding $H^k(\Omega) \subset \mathcal{C}(\bar{\Omega})$. In other words, for each $v \in H^k(\Omega)$ show that there is $w \in \mathcal{C}(\bar{\Omega})$ such that $v = w$ almost everywhere, i.e., $\|v - w\|_{L_2} = 0$. Hint: $\mathcal{C}^k(\bar{\Omega})$ is dense in $H^k(\Omega)$ and $\mathcal{C}(\bar{\Omega})$ is a Banach space.

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