The Equivariant Cuntz semigroup

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The Cuntz semigroup

Definition

Let A be a C*-algebra and let $a, b \in A_+$. We write $a \precsim b$ if there is $(d_n)_{n\in\mathbb{N}}$ in A such that

$$\lim_{n\to\infty} \|d_n b d_n^* - a\| = 0.$$

Write $a \sim b$ if $a \precsim b$ and $b \precsim a$.
The Cuntz semigroup of A, denoted by $\mathrm{Cu}(A)$, is defined as the set of Cuntz equivalence classes of positive elements of $A \otimes \mathcal{K}$.

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One shows that $\mathrm{Cu}(A)$ is a partially ordered abelian semigroup, and that $A \mapsto \mathrm{Cu}(A)$ is a functor from the category of C^*-algebras to a certain category of such semigroups.

The Cuntz semigroup

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Theorem (Robert, 2010)

Algebras in $\mathcal{R}$ are classified by their Cuntz semigroup.

The Equivariant Cuntz semigroup

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Let G be a compact group. Its *representation semiring* $\mathrm{Cu}(G)$ is the set of all unitary equivalence classes of unitary representations of G on separable Hilbert spaces. It is a semiring under direct sum and tensor product.

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We take $\mathbb{N} = \{0, 1, \ldots\}$ and $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

Example

Suppose that G is abelian. Then $\mathrm{Cu}(G) = \overline{\mathbb{N}}[\widehat{G}]$.

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Recall: $\mathrm{Cu}(G)$ is the set of equivalence classes of unitary representations of G.

Definition

Let G be a compact group, let A be a C^*-algebra and let $\alpha\colon G \to \mathrm{Aut}(A)$. The *equivariant Cuntz semigroup* $\mathrm{Cu}^G(A,\alpha)$ is defined using G-invariant positive elements in $A \otimes \mathcal{K}(\mathcal{H}_\pi)$, where π ranges over all unitary representations of G, and $A \otimes \mathcal{K}(\mathcal{H}_\pi)$ has the diagonal action of G.

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Cu^G looks like equivariant K-theory on the surface, but it is harder to work with.

The Equivariant Cuntz semigroup

Recall $\mathrm{Cu}(G) = \overline{\mathbb{N}}[\widehat{G}]$ when G is compact abelian.

Julg's Theorem for Cu^G

There is a natural isomorphism

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as semigroups.
When G is abelian, the $\mathrm{Cu}(G)$-semimodule structure is easy to describe: an element $\chi \in \widehat{G}$ acts via

$$\chi \cdot s = \mathrm{Cu}(\widehat{\alpha}_\chi)(s)$$

for $s \in \mathrm{Cu}(A \rtimes_\alpha G)$.

Cu^G as an invariant for group actions

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Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. An α-*cocycle* is a strongly continuous function $\omega\colon G \to \mathcal{U}(A)$ such that $\omega_{gh} = \omega_g \alpha_g(\omega_h)$ for all $g, h \in G$.
In this case, $\alpha^\omega\colon G \to \mathrm{Aut}(A)$ given by $\alpha^\omega_g = \mathrm{Ad}(\omega_g) \circ \alpha_g$ is also a continuous action.

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Theorem

There is a natural isomorphism $\mathrm{Cu}^G(A, \alpha) \cong \mathrm{Cu}^G(A, \alpha^\omega)$ as $\mathrm{Cu}(G)$-semimodules.

Cu^G does not distinguish cocycle conjugate actions.

Classification of Rokhlin actions

Definition

If G is finite, $\alpha\colon G \to \mathrm{Aut}(A)$ has the *Rokhlin property* if for every $\varepsilon > 0$ and every finite subset $F \subseteq A$, there exist projections $e_g \in A$ for $g \in G$ such that

1. $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$,
2. $\sum_{g\in G} e_g = 1$,
3. $\|e_g a - ae_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.

The Rokhlin property is the main hypothesis in most classification theorems for actions, and we will add one more to the list using the (equivariant) Cuntz semigroup.

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Recall: algebras in $\mathcal{R}$ are direct limits of 1-dim NCCW-complexes with trivial K_1-group, and Cu classifies algebras in $\mathcal{R}$.

Classification of Rokhlin actions on $\mathcal{R}$

Theorem

Let G be finite, let $A, B \in \mathcal{R}$ and let α and β be actions on A and B with the Rokhlin property. For every Cu-morphism

$$\rho\colon (\mathrm{Cu}(A), \mathrm{Cu}(\alpha)) \to (\mathrm{Cu}(B), \mathrm{Cu}(\beta)) \ \text{ with } [1_A] \mapsto [1_B]$$

there is $\phi\colon (A, \alpha) \to (B, \beta)$ equivariant which lifts ρ.

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$$\mathrm{Cu}^{\widehat{G}}(A \rtimes_\alpha G, \widehat{\alpha}) \to \mathrm{Cu}^{\widehat{G}}(B \rtimes_\beta G, \widehat{\beta})$$

preserving the unit can be lifted to equivariant homomorphisms. (The $\mathrm{Cu}(\widehat{G})$-semimodule structure is crucial.)

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Note: there is no cocycle.

Classification of locally representable actions on $\mathcal{R}$

We now turn to a different, though related, class of actions.

Definition

Let $A \in \mathcal{R}$, and write it $A = \varinjlim A_n$ as in the definition. If G is finite, an action $\alpha\colon G \to \mathrm{Aut}(A)$ will be called *locally representable* if it is the direct limit of inner actions on A_n.

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These actions can also be classified.

Theorem

Let G be finite abelian, let $A, B \in \mathcal{R}$ and let α and β be locally representable actions on A and B. For every $\mathrm{Cu}(G)$-semimodule morphism

$$\rho\colon \mathrm{Cu}^G(A,\alpha) \to \mathrm{Cu}^G(B,\beta) \ \text{ with } [1_A] \mapsto [1_B]$$

there are a β-cocycle ω and $\phi\colon (A,\alpha) \to (B,\beta^\omega)$ lifting ρ.

Theorem (continuation)

Let G be finite abelian, let $A, B \in \mathcal{R}$ and let α and β be locally representable actions on A and B. For every $\mathrm{Cu}(G)$-morphism

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- If ρ is invertible, then ϕ can be chosen to be invertible.
- ω is trivial iff $\rho([e_\alpha]) = [e_\beta]$.

Here e_α is the projection $e_\alpha = \frac{1}{|G|}\sum_{g\in G} u_g$ in the crossed product $A \rtimes_\alpha G$, and similarly for e_β.

Thank you.